Numerical solution of elliptic problems with singularities

Modeling second project
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1 Introduction

Elliptic partial differential equations (PDEs) play an important role to simulate physical phenomena, Poisson equation is a known elliptic problem, it can describe the potential field caused by a given electric charge or mass density distribution or other phenomena. Most of the time it is hard or impossible to find the analytic solution of this problem, and some methods were invented to approximate the solution of the PDEs, with discretization schemes or other technique methods. Our focus is based on finding the numerical solution using different finite difference schemes. first we determine the gradient in 3D for Poisson problem, then we present the approximate solution of 2D case on a so called L-shape d domain, where we are trying to understand the behavior of the problem at the singularity.

2 Approximate the modulus of the gradient in a regular domain

2.1 Estimation of the gradient of the Numerical solution in a 2D regular domain

To get a good approximation for the gradient we have to use different schemes corresponding to the position of the discretized points on in the domain.

2.1.1 Finite difference method for the approximate solution

Consider Poisson problem with Dirichlet boundary condition

$$\begin{cases} -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y) & x, y \in \Omega \\ u(x,y) = 0 & x, y \in \partial\Omega \end{cases}$$

Let Ω be a rectangle $\Omega; = (0,a) \times (0,b)$ and let us denote by u the approximate solution to the discretization points

$$u(x_i, y_j) = u(ih_1, jh_2)$$

where $h_1 = \frac{1}{N_1+1}$, $h_2 = h_1 = \frac{1}{N_2+1}$ are step sizes and N_1, N_2 are total number of discretization points in the direction of $[0,a]$ and $[0,b]$ respectively.
Consider the following different schemes to approximate derivatives:

1.Forward difference scheme it approximates the first derivative of u at x

$$u_i' = \frac{u_{i+1} - u_i}{h}$$

2.Backward difference scheme

$$u_i' = \frac{u_i - u_{i-1}}{h}$$

3.Center difference scheme

$$u'_i = \frac{u_{i+1} - u_{i-1}}{2h}$$

4.Second order finite difference scheme in 1D:

$$u''_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

this is extended in 2D for our problem as

$$-\Delta_h u = \frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{h_1^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i-1,j+1}}{h_2^2} \tag{1}$$

if we take into consideration that $h_1 = h_2$ then the matrix A which corresponds to the scheme is as follows: let

$$B = \begin{pmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{pmatrix}$$

and
$A = triblockdiag(I, B, I)$, that is,

$$A = \begin{pmatrix} B & -I & & & & \\ -I & B & -I & & & \\ & -I & B & -I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & B & -I \\ & & & & -I & B \end{pmatrix} \in \mathbb{R}^{N_1 \times N_1} \times \mathbb{R}^{N_2 \times N_2} \tag{2}$$

then the form of the problem will be $\Delta\tilde{u} = \frac{1}{h^2} * A * \tilde{u} \qquad (\tilde{u} \in \mathbb{R}^{N_1 \times N_2})$.

2.1.2 Numerical approximation of the gradient

the basic idea of approximating the modulus of gradient is that we have first to estimate the solution u, then we can use the different schemes which we mentioned before to calculate the approximation of $|\nabla u|$. In 2D case we used three different schemes concerning to the points in Ω and the points on the boundary and the points at the corners. We use the center scheme to estimate the partial derivatives of u inside the domain

$$\partial_x u_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$\partial_y u_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2h}$$

where $(x_i, y_j) \in \Omega$.

The forward and backward schemes are used on the edges depending on the position of the points.

$$\partial_x u_{0,j} = \frac{u_{1,j} - u_{0,j}}{h} \qquad (forward)$$

$$\partial_x u_{N_1+1,j} = \frac{u_{N_1+1,j} - u_{N_1,j}}{h} \qquad (backward)$$

then

$$|\nabla u| = \sqrt{\partial_x u_{i,j}^2 + \partial_y u_{i,j}^2}$$

For the calculation of the corners we don't calculate the partial derivatives at these points but we just calculate the diagonal derivative at each point related to its position.
The next figure illustrates how the gradient is approximated at each point, in other words the arrows show which scheme is used, based on the mesh of the rectangle $[0, a] \times [0, b]$

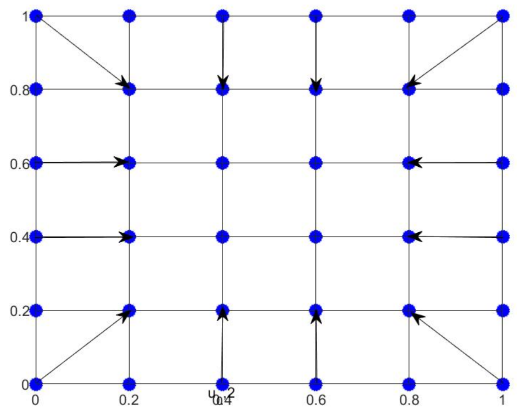

Figure 1: Discretization of Ω and the schemes which are used to approximate the gradient

The next two figures show us how the approximation of $|\nabla u|$ is close to the analytic value when u is known.

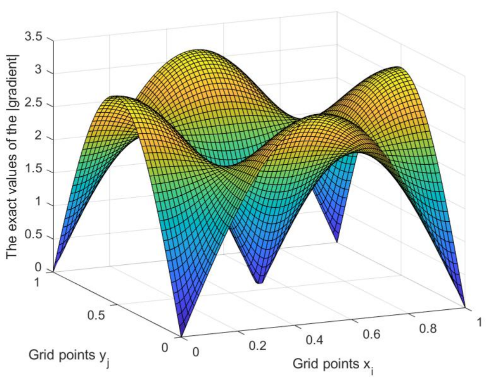

Figure 2: The analytic solution of the $|\nabla u|$

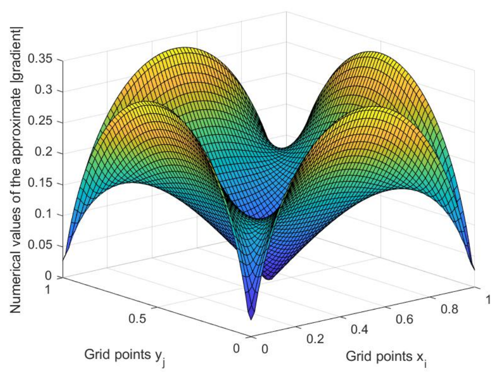

Figure 3: The approximate solution of the $|\nabla u|$

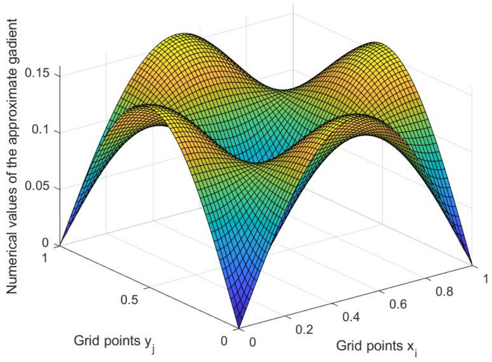

Figure 4: The approximate solution of the $|\nabla u|$ with different schemes

2.2 Numerical solution to estimate the gradient in 3D regular domain (cube)

Consider the Poisson problem with Dirichlet boundary conditions

$$\begin{cases} \Delta u = f & x, y, z \in \Omega \\ u(x, y, z) = 0 & x, y, z \in \partial\Omega \end{cases}$$

The cube $[0, a] \times [0, b] \times [0, c]$ is divided with a uniform step size h. Let us take the simple case when a=b=c=1 then the difference schema of our problem will be

$$\Delta u(ih, jh, kh) \approx \frac{u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k-1} + u_{i,j,k+1} - 6u_{i,j}}{h^2}$$

where $i = 1, \ldots, N \quad j = 1, \ldots, N, \quad k = 1, \ldots, N$, and N is the total number of the discretization points at each line.

The matrix form of the corresponding difference schema in 3D is given

$$B := \text{tridiag}(-1, 6, -1) \in \mathbb{R}^{N \times N}$$
$$C := \text{blocktridiag}(-I, B, -I) \in \mathbb{R}^{N^2 \times N^2}$$
$$A_h = \frac{1}{h^2} \text{ blocktridiag } (-I, C, -I) \in \mathbb{R}^{N^3 \times N^3}$$

$$A_h = \begin{pmatrix} C & -I & & & & \\ -I & C & -I & & & \\ & -I & C & -I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & C & -I \\ & & & & -I & C \end{pmatrix}$$

where

$$C = \begin{pmatrix} B & -I & & & & \\ -I & B & -I & & & \\ & -I & B & -I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & B & -I \\ & & & & -I & B \end{pmatrix}$$

2.2.1 The numerical solution for the gradient in cubic domain

considering the same problem, for approximating the mesh points we consider different schemes. in order to find the estimation of $|\nabla u|$ at the corners we use the following schemes

$$|\nabla u(x_0, y_0, z_0)| = \frac{|u(x_1, y_1, z_1) - u(x_0, y_0, z_0)|}{\sqrt{3}h}$$

$$|\nabla u(x_{N+1}, y_{N+1}, z_{N+1})| = \frac{|u(x_{N+1}, y_{N+1}, z_{N+1}) - u(x_N, y_N, z_N)|}{\sqrt{3}h}$$

we do apply similar schemes to the other corners. The forward and backward schemes are used to estimate the $|\nabla u|$ in the faces and in each face of the domain we have $|\nabla u|$ is equal to the partial derivative of that corresponding fixed axes and the other partial derivatives are zeros because of the Dirichlet boundary conditions. Example for illustration, consider the face where $y = 0$

$$|\nabla u(x_i, y_0, z_k)| = |\partial_y u(x_i, y_0, z_k)| = \frac{|u(x_i, y_1, z_k) - u(x_i, y_0, z_k)|}{h}$$

another implementation for another face where $x = 0$

$$|\nabla u(x_0, y_j, z_k)| = |\partial_x u(x_0, y_j, z_k)| = \frac{|u(x_1, y_j, z_k) - u(x_0, y_j, z_k)|}{h}$$

The estimation on the edges are taken the central second order scheme and the first order forward and backward schemes, while inside the cube the only used scheme is the central scheme. To see the plotting of the $|\nabla u|$ approximation we take the r.h.s of the problem as

$$f(x, y, z) = d * e^{-R*((x-a)^2+(y-b)^2+(z-c)^2)}$$

where a, b, c, d, R are constants.

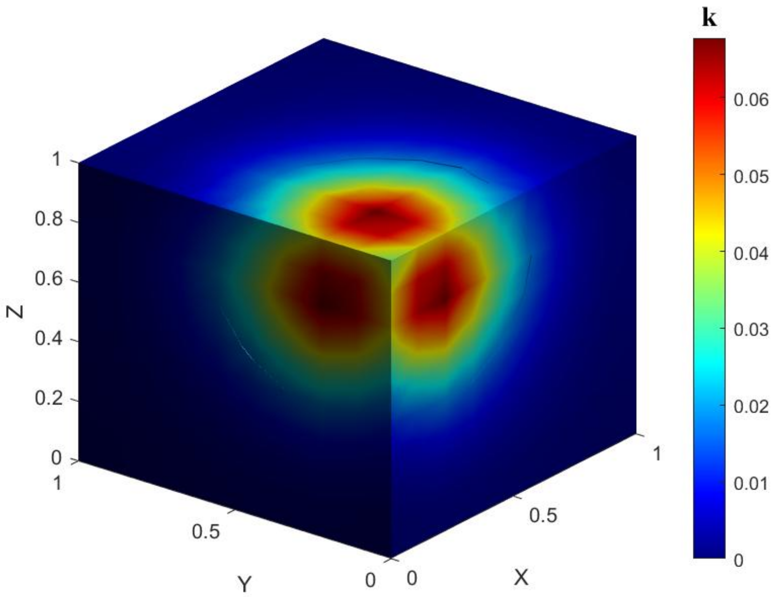

Figure 5: The approximate solution of the $|\nabla u|$ in cubic domain

3 Approximate the numerical solution in 2D L-shape domain

Consider the same problem with right hand side equals to one, and use the five point stencil method 1 to make a good approximation for the the solution u in the L-shape domain, the first thing we should be careful with is the changing of boundary conditions, we should take into account also the effect of the singularity in our case is the origin. Let us take a uniform step size h and the domain $\Omega = (-1,1) \times (-1,1) \backslash \{(0,1) \times (0,1)\}$, then the problem is given as

$$\begin{cases} \Delta u_{|\Omega} = f \\ u_{|\partial\Omega} = 0 \end{cases}$$

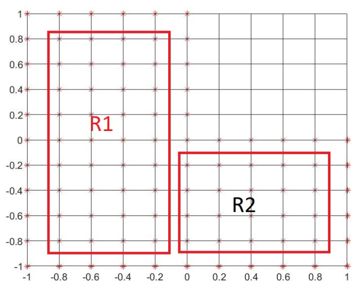

Figure 6: The Domain Ω.

As we can see from Figure 6 we will make an approximation for two rectangles, the matrix form A which corresponds to the approximate scheme will be given as same as in 2, but there is just difference in the dimension of the matrix where $N1$ here describes the number of discretizations correspond to the $(-1, 0)$ line and $N2$ presents the discretizations on the vertical line in Ω the same criteria we do with the R2 where N1 presents the number of discretizations of the horizontal line $[0, 1)$ and N2 corresponds to the vertical line $(-1, 0)$, everything is clear so far except the line which is located between $R1$ and $R2$, where we have to take into account that when we would like to approximate u_{N1j} j=N+1..2*N+2 (we choose up-down numbering) and the same for R2 we have to deal with u_{1j} j=N+1..2*N+2. Our idea is to merge the two matrices in one block matrix and after that we modify on the two matrices which are located above and under the diagonal. The next figure shows the plotting of the approximate solution on the L-shape domain $\bar{\Omega}$.

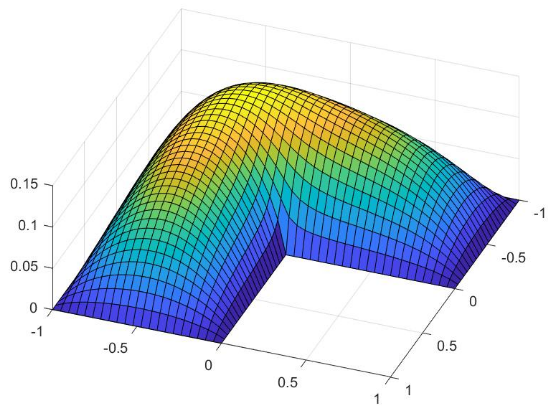

Figure 7: The Domain Ω.

4 Numerical solution in 3D L-shape domain

As we have seen for approximating the solution in the 2D L-shape domain using the finite difference method it is a bit difficult to deal with the singularity and the lines between the fractured rectangles. We did that to get a better approximation taking into account the boundary conditions, however when it comes to work on a 3D L-shape domain the situation gets more complex, it can happen that we can take every face in it and we treat it as 2D case, but it will take much time to solve that and maybe it will not give us proper approximation that is why we have to find other methods, such that they will give us better accuracy near the singularities. One of those methods is finite element method which takes some type of approximation and try to approximate in the neighbourhood of singularities.

5 Conclusion

This work demonstrates that the finite difference method can make good approximation to the solution, however sometimes it can be challenging when we deal with the boundary conditions, and its convergence can not be much effective that why a lot of methods have invented to deal with these problems, such as the finite element methods and this problem can be very challenging if we work on an irregular domain e.g we have studied in this paper about the L shape domain and its difficulties when it comes to approximate the solution at the singularity.

References

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