Numerical solution of elliptic problems with singularities

TAKI EDDINE DJEBBAR

Supervisor: Prof. János Karátson

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Real life applications

- mass density distribution
- electric charge
- wave equation (e.g water waves, sound waves)

Problem description

Let us consider the Poisson problem

$$-\Delta u = f \quad \Leftrightarrow \quad \begin{cases} -div\, B = f \\ B = \nabla u \end{cases}$$

The problem will also satisfy the boundary condition $\partial\Omega$

$$\begin{cases} -\Delta u = f \\ u_{|\partial\Omega} = g \end{cases}$$

Approximate the modulus of the gradient in a regular domain

2D case: let $\Omega = (0, a) \times (0, b)$, and we take $h_1 = \frac{1}{N_1+1}$,

$h_2 = \frac{1}{N_2+1}$

Let us consider the Poisson problem with Dirichlet boundary condition

$$\begin{cases} -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y) & x, y \in \Omega \\ u(x, y) = 0 & x, y \in \partial\Omega \end{cases}$$

The approximate solution u

First we have to estimate the solution u using the Second order finite difference scheme

$$-\Delta_h u = \frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{h_1^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i-1,j+1}}{h_2^2}$$

Matrix form of the approximate solution u

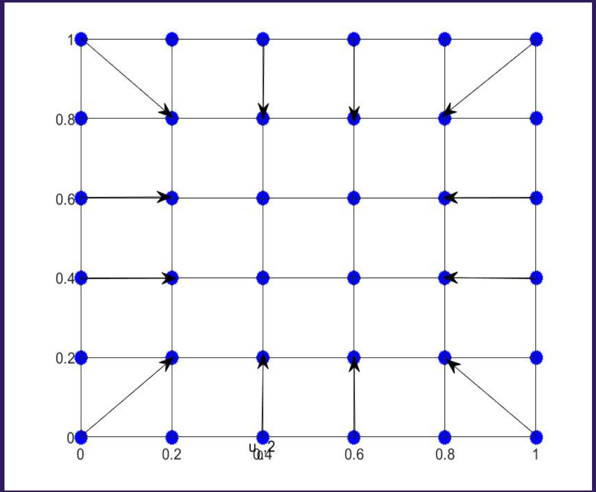
$$B = \begin{pmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{pmatrix}$$

$A = triblockdiag(I, B, I)$

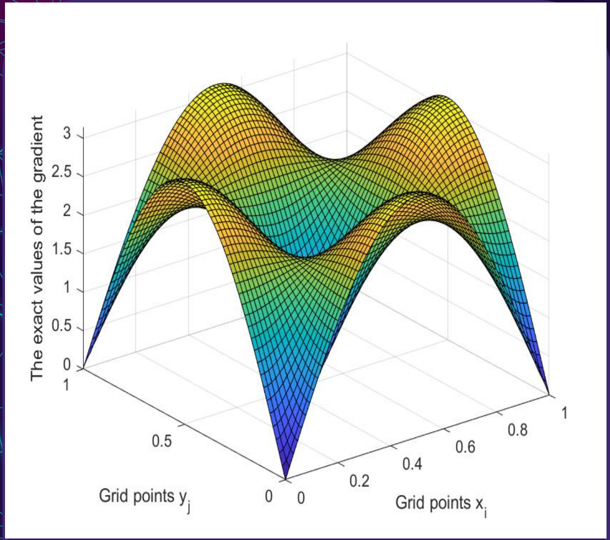
$$A = \begin{pmatrix} B & -I & & & & \\ -I & B & -I & & & \\ & -I & B & -I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & B & -I \\ & & & & -I & B \end{pmatrix} \in \mathbb{R}^{N_1 \times N_1} \times \mathbb{R}^{N_2 \times N_2}$$

The approximate of modulus of the gradient

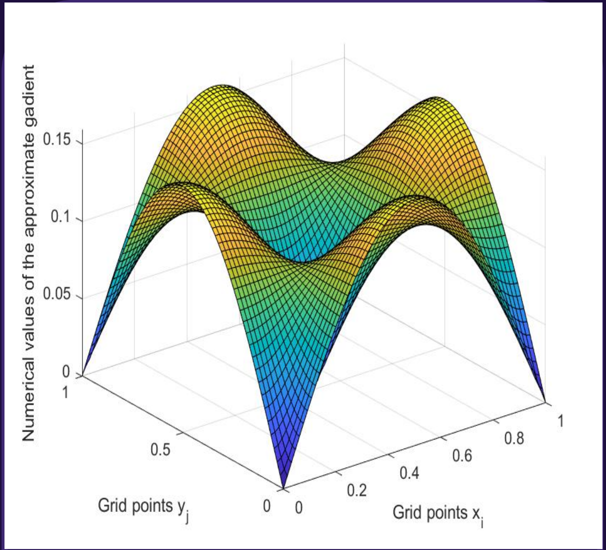
After finding the approximate solution u, know we can approximate the modulus of the gradient as follows



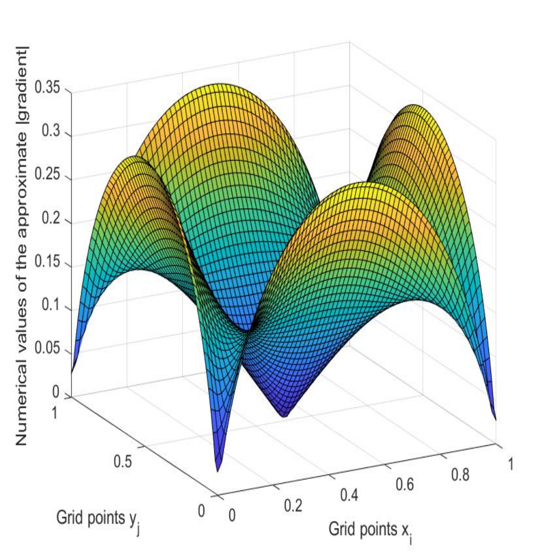
The results of 2D with $f(x,y) = 2\pi^2 sin(x) sin(y)$



The analytic solution of the |∇u| with N1 and N2 =64



The approximate solution of the |∇u| with N1 and N2 =64



The approximate solution of the |∇u| with N1 and N2 =64 using the schemes in the last figure

Numerical solution to estimate the gradient in 3D regular domain (cube)

Let let $\Omega = (0,1) \times (0,1) \times (0,1)$, and we take $h_1 = h_2 = h_3 = \frac{1}{N+1}$.

Consider the Poisson problem

$$\begin{cases} -\Delta u = f(x,y,z) & x,y,z \in \Omega \\ u(x,y,z) = 0 & x,y,z \in \partial\Omega \end{cases}$$

To get the approximate solution of this problem we use the scheme

$$\Delta u(ih, jh, kh) \approx \frac{u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k-1} + u_{i,j,k+1} - 6u_{i,j}}{h^2}$$

The matrix form

$B = tridiag(-1,6,-1) \in \mathbb{R}^{N \times N}$

$C = triblockdiag(-I, B, -I) \in \mathbb{R}^{N^2 \times N^2}$

$A_h = triblockdiag(-I, B, -I) \in \mathbb{R}^{N^3 \times N^3}$

$$A_h = \begin{pmatrix} C & -I & & & & \\ -I & C & -I & & & \\ & -I & C & -I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & C & -I \\ & & & & -I & C \end{pmatrix}$$

Where the matrix C is

$$C = \begin{pmatrix} B & -I & & & & \\ -I & B & -I & & & \\ & -I & B & -I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & B & -I \\ & & & & -I & B \end{pmatrix}$$

The numerical estimation for the gradient in cubic domain

In order to estimate the modulus of the gradient we will use the approximate solution u at each point in the mesh points and different schemes

$$|\nabla u(x_0, y_0, z_0)| = \frac{|u(x_1, y_1, z_1) - u(x_0, y_0, z_0)|}{\sqrt{3}h}$$

(Estimate at the corner (0,0,0))

$$|\nabla u(x_0, y_j, z_k)| = |\partial_x u(x_0, y_j, z_k)| = \frac{|u(x_1, y_j, z_k) - u(x_0, y_j, z_k)|}{h}$$

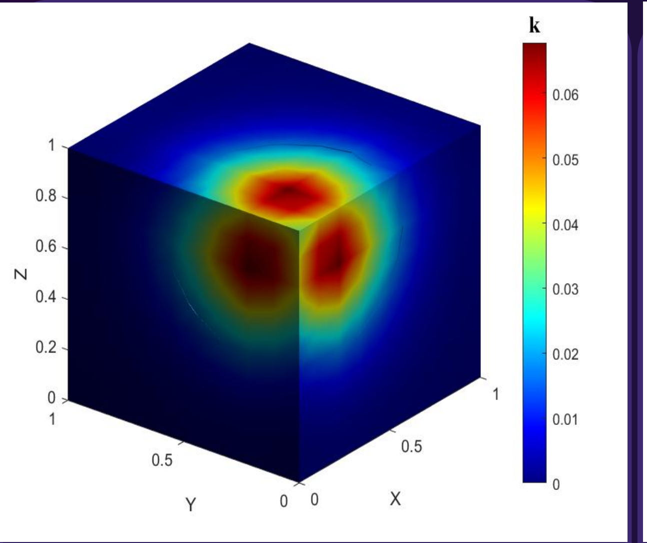
Estimate the $|\nabla u|$ on the faces

For estimating the points in Ω we use the central difference scheme to estimate first the partial derivatives after that estimate $|\nabla u|$.

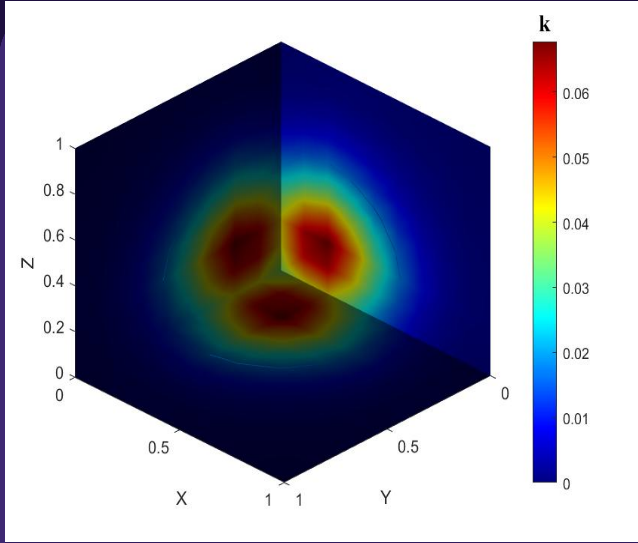
The numerical results result for estimate the modulus of the gradient

Let $f(x,y,z) = d \cdot e^{-R\left((x-a)^2+(y-b)^2+(z-c)^2\right)}$ a, b, c, d, R *constants*.

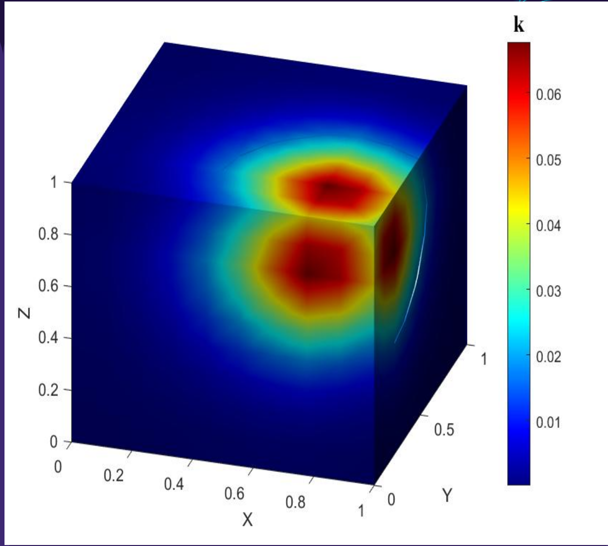
The plot of the modulus



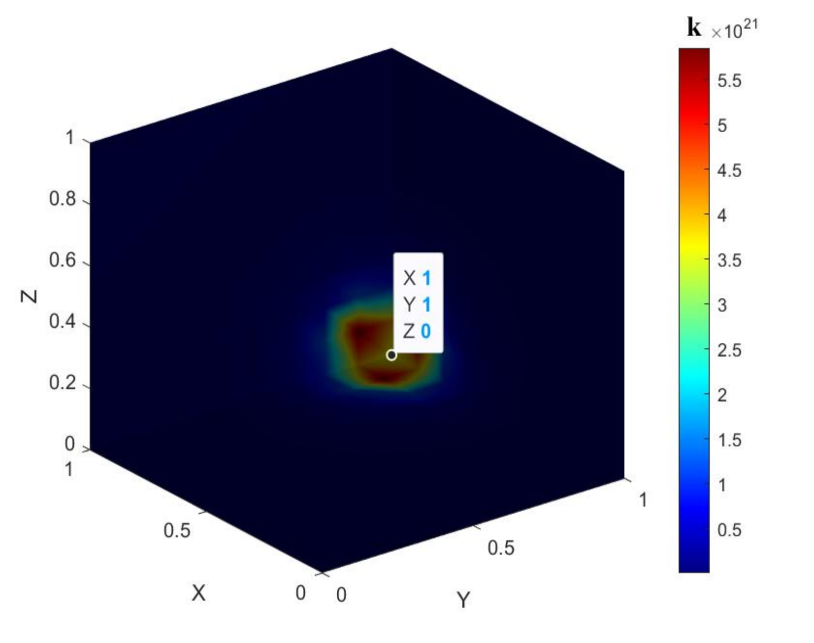
$R = 8, a = 0.01, b = 0.01, c = 0.99, d = 2$



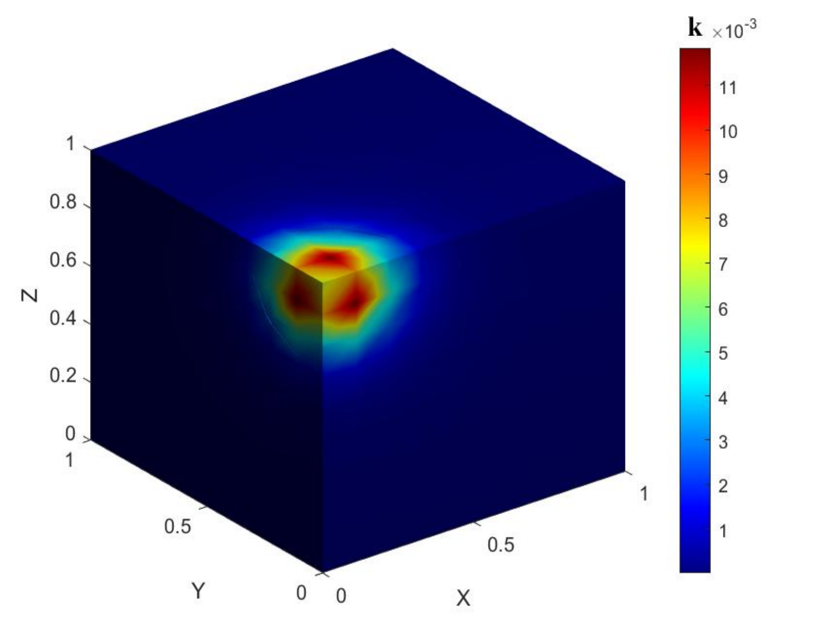
$R = 8, a = 0.01, b = 0.01, c = 0.01, d = 2$



$R = 8, a = 0.99, b = 0.01, c = 0.99, d = 2$



$R = -23, a = 0.01, b = 0.01, c = 0.99, d = 2$



$R = 45, a = 0.01, b = 0.01, c = 0.99, d = 2$

Approximate the numerical solution in 2D L-shape domain

Consider the Poisson problem with f=1 in $\Omega = (-1,1) \times (-1,1) \setminus \{(0,1) \times (0,1)\}$

The solution u will be approximated using the five stencil approximation method and based on the sppliting the domain into two parts

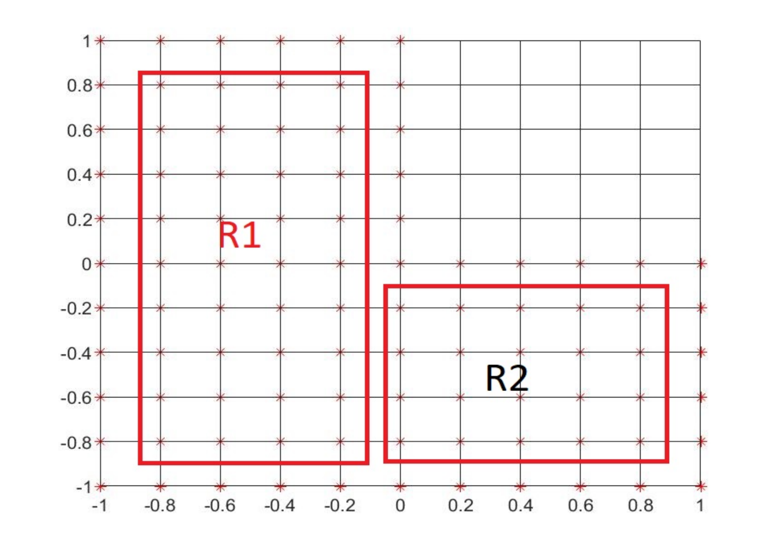


Figure01: mesh grid points in L-shape domain

Numerical results

```
T1=N*(N+2):N:(2*N+1)*N;
T2=(2*N+1)*N+1:N+1:NN(1);    % for calling the indices componentwise
for i=1:length(T1)
A_h(T1(i),T2(i))=-1;     % modify A_h in order to approximate the line S2 and S3
A_h(T2(i),T1(i))=-1;
end
A_h=(1/(h^2))*A_h;
[x1,y1]=meshgrid(-1:h:0,1:-h:-1);
[x2,y2]=meshgrid(0:h:1,0:-h:-1);
X=[x1;x2];
Y=[y1;y2];
App_sol=[Y12;Y3];
figure
surf(X,Y,App_sol)
```

The plotting of the numerical solution u

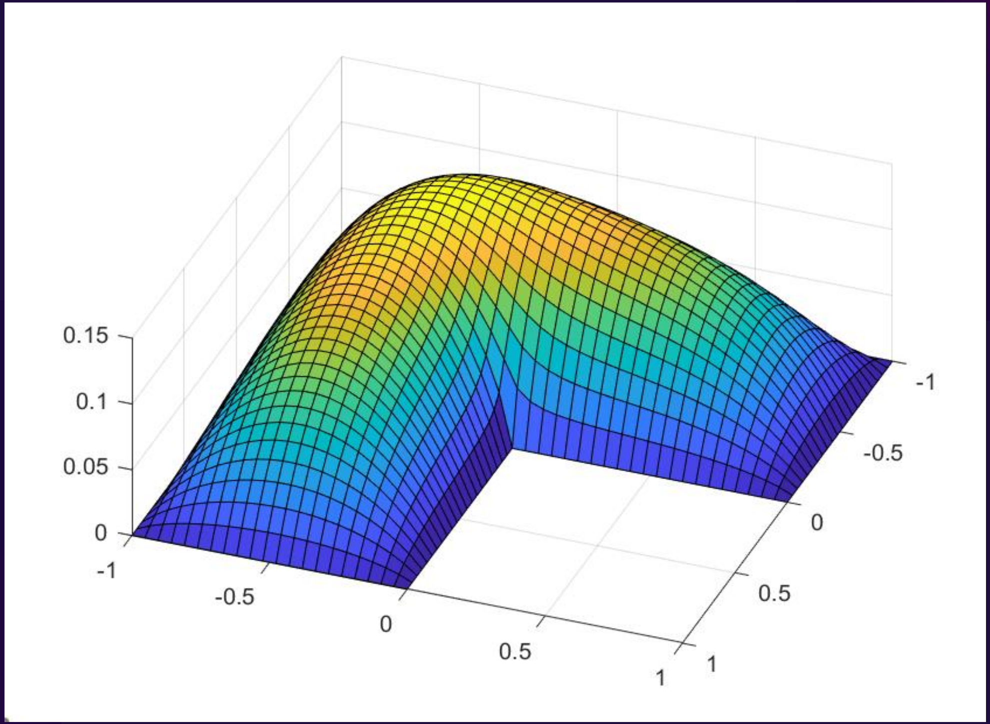


Figure02: Approximate solution u in L-shape domain

Singular element

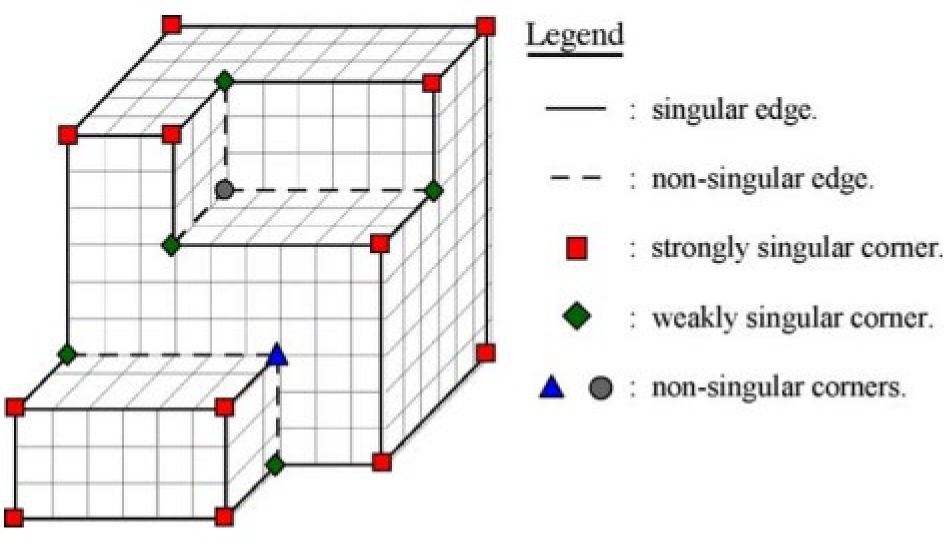


Figure03: types of singularities

Thank you for your attention