Numerical modelling of disease propagation

Math Project II.

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1 Introduction

In 2020, Yang and Wang proposed the following model to investigate the epidemic period of COVID-19 in Wuhan from January 23, 2020 to February 10, 2020[1]:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \beta_E SE - \beta_I SI - \beta_V SV - \mu S \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \beta_E SE + \beta_I SI + \beta_V SV - (\alpha + \mu)E \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \alpha E - (w + \gamma + \mu)I \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I - \mu R \\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \xi_1 E + \xi_2 I - \sigma V
\end{aligned} \tag{1}$$

Parameters	
Λ	Population influx
μ	Natural death rate
w	Disease induced death rate
$1/\alpha$	Mean incubation period
γ	Recovery rate
β_I	Transmission rate by infected individual
β_E	Transmission rate by exposed individual
β_V	Transmission rate by the environmental reservoir
ξ_1	Rate of the exposed individuals contributing the virus to the environment
ξ_2	Rate of the infected individuals contributing the virus to the environment
σ	Rate of (natural and artificial) removal of the virus from the environment

where S, E, I, R are the number of susceptible, exposed (infectious but not yet symptomatic), infected (infectious and symptomatic) and recovered, respectively. All the parameters are non-negative. The core of the model is the usual SEIR model with variables (S,E,I,R), the main changes are the mass-action incidence $\beta_V SV$ in the compartment S and E and the new compartment V with its own dynamics. V represents the environmental reservoir and is integrated to the usual SEIR model to include the possibility that a susceptible individual may acquire the disease through the environment and not directly by susceptible-infectious contacts. Many cholera models also have similar explicit environmental compartments[2].

We are mainly interested in the numerical modeling aspect of this model (i.e. which properties of the model are inherited after different discretizations). The discretization of the different continuous epidemiological models are inevitable if we want to solve them numerically. One of the simplest one-step numerical method is the explicit Euler method: $x_{n+1} = x_n + hf(t_n, x_n)$, where $h > 0$ is the (time) discretization step-size. By discretizing (1) by the explicit Euler method, we get the following (discrete) system:

$$\begin{aligned}
s_{n+1} &= s_n + h(\Lambda - \beta_E s_n e_n - \beta_I s_n i_n - \beta_V s_n v_n - \mu s_n) \\
e_{n+1} &= e_n + h(\beta_E s_n e_n + \beta_I s_n i_n + \beta_V s_n v_n - (\alpha + \mu)e_n) \\
i_{n+1} &= i_n + h(\alpha e_n - (w + \gamma + \mu)i_n) \\
r_{n+1} &= r_n + h(\lambda i_n - \mu r_n) \\
v_{n+1} &= v_n + h(\xi_1 e_n + \xi_2 i_n - \sigma v_n)
\end{aligned} \tag{2}$$

where the lower case variables s, e, i, r, v have the same meaning as their uppercase equivalents. s_0, e_0, i_0, r_0, v_0 are given ($s_0 = S(0)$, etc.) and non-negative.

2 Property preservation of the discrete model

We can look at system (2) as a discrete autonomous system and we can ask similar questions as in the continuous case (1). For a discrete (autonomus) dynamical system $x_{n+1} = f(x_n)$, we call x^* an equlibrium point if $x^* = f(x^*)$ (i.e. constant solution). It can be shown that $x^* \in \mathbb{R}^5$ is an equlibrium point of system (1) if and only if it is an equlibrium point of (2), more then that, it is true for the explicit (and implicit) Euler discretization method independently of the considered system, because if we have a continuous system $\dot{x} = f(x)$ with x^* such that $f(x^*) = 0$, then after the discretization we get the discrete system $x_{n+1} = x_n + hf(x_n)$ (for the implicit method $x_{n+1} = x_n + hf(x_{n+1})$) with equilibrium points $x^* = x^* + f(x^*)$. Note that this is not true in general for one-step numerical methods, so-called *spurious equilibrium points* can appear which of them are not equilibrium points of the continuous system and their values depends on h and for $h \to 0$ their limits are the real equilibrium points[3].

The two equilibria of the system (2) are $\mathscr{E}_0 = (S_0, 0, 0, 0, 0)^T$ where $S_0 := \frac{\Lambda}{\mu}$ and $\mathscr{E}_1 = (s^*, e^*, i^*, r^*, v^*)^T$, where

$$\begin{aligned}
s^* &= \frac{\alpha+\mu}{\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_v} \\
e^* &= \frac{\Lambda}{\alpha+\mu} - \frac{\mu}{\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_V} \\
i^* &= \frac{\Lambda\alpha}{w_1(\alpha+\mu)} - \frac{\alpha\mu}{w_1(\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_v)} \\
r^* &= \frac{\gamma\alpha\Lambda}{\mu w_1(\alpha+\mu)} - \frac{\gamma\alpha}{w_1(\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_V)} \\
v^* &= \frac{\Lambda c}{\alpha+\mu} - \frac{c\mu}{\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_v}
\end{aligned}$$

and $w_1 = \gamma+\mu+w$, $c = \frac{\xi_1 w_1 + \xi_2\alpha}{w_1\sigma}$. Note that s^* is always positive, whereas e^*, i^*, r^*, v^* are positive if and only if $\mathscr{R}_0 > 1$, where $\mathscr{R}_0$ is the basic reproduction number:

$$\mathscr{R}_0 = \frac{\beta_E S_0}{\alpha+\mu} + \frac{\beta_I S_0 \alpha}{(\alpha+\mu)(w+\gamma+\mu)} + \frac{\beta_V S_0(\alpha\xi_2 + (w+\gamma+\mu)\xi_1)}{(\alpha+\mu)(w+\gamma+\mu)\sigma}.$$

the number of secondary infections produced by an infected individual in a fully susceptible population (threshold parameter for the invasion of a disease organism into the population)[4]. While for the continuous system (1)

$$\Omega = \left\{ (S, E, I, R, V) \in \mathbb{R}^5_+ : S+E+I+R \le \frac{\Lambda}{\mu}, 0 \le V \le \frac{(\xi_1+\xi_2)\Lambda}{\mu\sigma} \right\} \tag{3}$$

is positively invariant region[1]. For the discrete system (2) we have the following theorem about the non-negativity and the boundedness:

Theorem 1. *The discretized system (2) is positively invariant in Ω if $h \le min\{\frac{1}{\mu+(\beta_e+\beta_i+\beta_v)s_0}, \frac{1}{\alpha+\mu}, \frac{1}{w+\gamma+\mu}\}$ and $h < \frac{1}{\sigma}$.*

Proof. Let n_n denote the total population at time n: $n_n = s_n + e_n + i_n + r_n$. Suppose that $(s_{n-1}, e_{n-1}, i_{n-1}, r_{n-1}, v_{n-1}) \in \Omega$, then by adding the first four equations of system (2):

$$n_n = n_{n-1} + h(\Lambda - \mu n_{n-1} - \lambda i_n) \le n_{n-1}(1-\mu h) + \Lambda h$$

By recursively using the inequality

$$\begin{aligned}
n_n &\le n_{n-1}(1-\mu h) + \Lambda h \le n_{n-2}(1-\mu h)^2 + h\Lambda(1-\mu h) + h\Lambda \le ... \\
&\le n_0(1-\mu h)^n + h\Lambda \sum_{k=0}^{n-1}(1-\mu h)^k = \frac{\Lambda}{\mu} + (n_0 - \frac{\Lambda}{\mu})(1-\mu h)^n
\end{aligned} \tag{4}$$

From where we can see that if $h < \frac{1}{\mu}$, then $n_0 \in [0, \frac{\lambda}{\mu}] \implies n_n \in [0, \frac{\Lambda}{\mu}]$ for all $n \in \mathbb{N}$. Similarly

$$\begin{aligned}
v_n &= (1-h\sigma)v_{n-1} + h(xi_1 e_{n-1} + \xi_2 i_{n-1}) \le (1-h\sigma)v_{n-1} + h(\xi_1+\xi_2)\frac{\Lambda}{\mu} \le \\
&\le \left(v_0 - \frac{\xi_1+\xi_2}{\sigma}\frac{\Lambda}{\mu}\right)(1-h\sigma)^n + \frac{\xi_1+\xi_2}{\sigma}\frac{\Lambda}{\mu}
\end{aligned} \tag{5}$$

From where we get that if $h < \frac{1}{\sigma}$ then $v_0 \in [0, \frac{\xi_1+\xi_2}{\sigma}\frac{\Lambda}{\mu}] \implies n_n \in [0, \frac{\xi_1+\xi_2}{\sigma}\frac{\Lambda}{\mu}]$ for all $n \in \mathbb{N}$. To get conditions on the positivity for each variable we use the same logic as in [5]. For the first variable, we want to show that $s_n \in [0, \frac{\lambda}{\mu}]$. From the first equation of (2):

$$s_n = s_{n-1} + h(\Lambda - \beta_E s_{n-1}e_{n-1} - \beta_I s_{n-1}i_{n-1} - \beta_V s_{n-1}v_{n-1} - \mu s_{n-1})$$

The inequality holds if and only if

$$s_{n-1} \geq -h(\Lambda - \beta_E s_{n-1}e_{n-1} - \beta_I s_{n-1}i_{n-1} - \beta_V s_{n-1}v_{n-1} - \mu s_{n-1}). \tag{6}$$

If

$$-(\Lambda - \beta_E s_{n-1}e_{n-1} - \beta_I s_{n-1}i_{n-1} - \beta_V s_{n-1}v_{n-1} - \mu s_{n-1}) \leq 0$$

than the inequality (6) holds for any step size. If

$$-h(\Lambda - \beta_E s_{n-1}e_{n-1} - \beta_I s_{n-1}i_{n-1} - \beta_V s_{n-1}v_{n-1} - \mu s_{n-1}) > 0$$

then we have to show that

$$h < \frac{s_{n-1}}{-\Lambda + \beta_E s_{n-1}e_{n-1} + \beta_I s_{n-1}i_{n-1} + \beta_V s_{n-1}v_{n-1} + \mu s_{n-1}} \tag{7}$$

for some h. From the inequality:

$$\begin{aligned} \frac{1}{\mu + (\beta_e + \beta_i + \beta_v)S_0} &= \frac{s_{n-1}}{s_{n-1}\mu + (\beta_e + \beta_i + \beta_v)S_0 s_{n-1}} \\ &\leq \frac{s_{n-1}}{-\Lambda + \beta_E s_{n-1}e_{n-1} + \beta_I s_{n-1}i_{n-1} + \beta_V s_{n-1}v_{n-1} + \mu s_{n-1}} \end{aligned} \tag{8}$$

So for any $h \leq \frac{1}{\mu+(\beta_e+\beta_i+\beta_v)S_0}$ the inequality (6) holds, i.e. $s_n \geq 0$.
For e_n, i_n, r_n, v_n the proof can be carried out similarly, but one gets simpler sufficient conditions for h because of the sign of the terms. □

Note that (8) is sufficient but not necessary condition, the non-negativity of (for example) s_{n+1} is fully determined by the condition (7). However, our numerical experiment shows that there are some necessary conditions for the non-negativity of the variables. This can be seen in fig: 1, where we used the same parameters and initial values as in [1] with $h = 2$. In this case $h > \frac{1}{\mu+(\beta_e+\beta_i+\beta_v)s_0}$, $\frac{1}{\alpha+\sigma} < h$ and $\frac{1}{w+\alpha+\mu} < h$ but $\frac{1}{\sigma} > h$. We can see that non-negativity does not hold for the variables e, i, v. We also see that (rapid) oscillations arises, just as in the case of the Dahlquist test equation for some h when we use the explicit Euler discretization scheme. Also note that from the inequality (7) and there equivalents for the other variables we can construct an adaptive explicit Euler method with variable step-size for which the non-negativity is preserved.

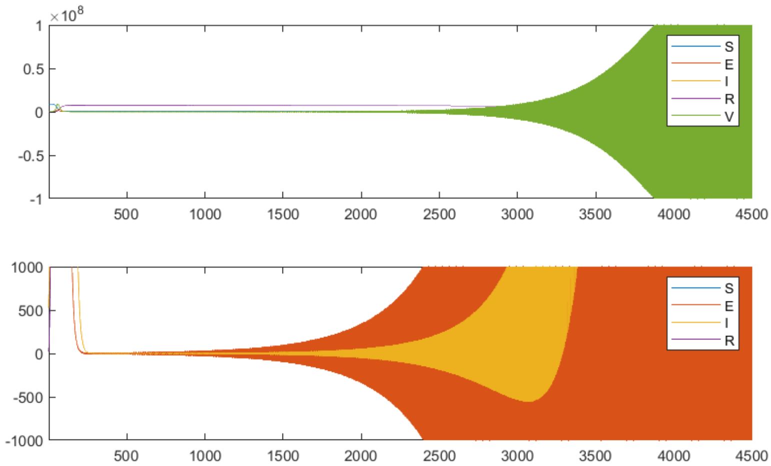

Figure 1: Negativity of the Explicit Euler method when $h = 2$. The lower figure is the same as the upper one, but the variable V is not shown, and it is also zoomed in.

For the continuous model (1) it was shown that in the case of $\mathscr{R}_0 < 1$, the boundary equilibrium is globally asymptotically stable on Ω and the endemic equilibrium is unstable, while in the case $\mathscr{R}_0 > 1$ the stability of the two equilibria changes[1]. In order to obtain some conditions for the stability of (2) at $\mathscr{E}_0$ one can linearize the system at $\mathscr{E}_0$. The characteristic polynomial of the Jacobian at $\mathscr{E}_0$ is:

$$F^*(\lambda) = (1 - h\mu - \lambda)^2 F(\lambda) = (1 - h\mu - \lambda)^2(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3)$$

where

$$b_1 = -3 - hs_1$$
$$b_2 = 3 + 2hs_1 + h^2 s_2$$
$$b_3 = -1 - hs_1 - h^2 s_2 + h^3 s_3$$

and s_1, s_2, s_3 are some functions of our parameters. Note that s_3 is always positive. The linearized system is asymptotically stable at the boundary equilibrium if all the eigenvalues of the Jacobian lie inside the unit disk. For this, we must have $h < \frac{1}{\mu}$, while for $F(\lambda)$, the Schur-Cohn criterium[6] gives necessary and sufficient conditions which are the following:

(i.) $F(1) > 0$

(ii.) $-F(-1) > 0$

(iii.) $|1 - b_3^2| > |b_2 - b_1 b_3|$

(i.) holds if and only if $\mathscr{R}_0 < 1$, while (ii.) gives a cubic polynomial inequality: $p(h) = h^3 - 2h^2\frac{s_2}{s_3} - 4h\frac{s_1}{s_3} - \frac{5}{s_3} < 0$. $p(0)$ is always negative because s_3 is always positive, so we can state that there exists an interval $(0, h^*)$, where (ii.) holds. For (iii.) the left hand-side is a sixth-order polynomial, while the right hand-side is a fourth-order polynomial with variable h. This does not hold unconditionally for any $(0, h^{**})$ interval in the sense that there exists $s_1, s_2 \in \mathbb{R}$ and $s_3 > 0$ (without thinking about how possible is to get that s_1, s_2, s_3) such that the inequality does not hold, but I was not able to give conditions for which (iii.) holds.

Another possibility for obtaining sufficient conditions for the stability of an equilibrium is by Gershgorin's disks. At the endemic equilibrium $\mathscr{E}_1$, the Jacobian $J(\mathscr{E}_1)$ is

$$\begin{pmatrix} 1 - h(\beta_e e^* + \beta_i i^* + \beta_v v^* + \mu) & -h\beta_e s^* & -h\beta_i s^* & 0 & -h\beta_v s^* \\ h(\beta_e e^* + \beta_i i^* + \beta_v v^*) & 1 + (h\beta_e s^* - (\alpha + \mu)) & h\beta_i s^* & 0 & h\beta_v s^* \\ 0 & h\alpha & 1 - h(w + \gamma + \mu) & 0 & 0 \\ 0 & 0 & h\gamma & 1 - h\mu & 0 \\ 0 & h\xi_1 & h\xi_2 & 0 & 1 - h\sigma \end{pmatrix} \tag{9}$$

We can see that $1 - h\mu$ is an eigenvalue, so we must have $h < \frac{2}{\mu}$. $\mathscr{R}_0 > 1$ is a necessary condition for the asymptotic stability, because if $\mathscr{R}_0 < 1$, then one of the eigenvalues of the matrix must lie on the $\mathbb{R} > 1$ half plane (because at the equilibria $\frac{\partial x_{n+1}}{\partial x_n} = I + h\frac{\partial f(x)}{\partial x}$ holds between the Jacobian of the continuous and the discrete system). We can give sufficient conditions on the stability by Gershgorin disks. Let A be a real quadratic matrix, then to have all of its eigenvalues in the unit disk, we must have $a_{ii} - R^i(A) > -1$ and $a_{ii} + R^i(A) < 1$ for all of its rows, where $R^i(A)$ is the i-th deleted absolute row sum of A. All of the diagonal elements of $J(\mathscr{E}_1)$ are smaller than 1, and it has a special structure such that the first condition depends on h, while the second condition is independent of h. With some tedious calculations it can be shown that if

$$h < min\left\{\frac{2}{\xi_1 + \xi_2 + \sigma}, \frac{2}{\alpha + w + \gamma + \mu}, \frac{2}{\mu}, \frac{2}{\alpha + \mathscr{R}_0\mu + \frac{\Lambda(\beta_e + \beta_i + \beta_v)}{\mu\mathscr{R}_0}}\right\}$$

and $\alpha < w + \gamma + \mu$, $\xi_1 + \xi_2 < \sigma$, $(\beta_e - \beta_i)\frac{\Lambda}{\mu\mathscr{R}_0} < \alpha + \mu\mathscr{R}_0$ and $\Lambda(\beta_e + \beta_i + \beta_v) < \mathscr{R}_0^2\mu^2$ then the discrete system (2) is (locally) asymptotically stable at the endemic equilibrium. If $\mathscr{R}_0 < 1$ then the discrete system (2) is unstable at the endemic equilibrium for all h. One can also get sufficient conditions by considering the i-th deleted absolute column sums, for which the sufficient conditions are

$$h < min\left\{\frac{2}{\mu}, \frac{2}{\mu(2\mathscr{R}_0 - 1)}, \frac{2}{2\alpha + \mu + \xi_1}, \frac{2}{w + 2\lambda + \mu + \xi_2 + 2\beta_i\frac{\Lambda}{\mu\mathscr{R}_0}}, \frac{2}{\sigma + 2\beta_v\frac{\Lambda}{\mu\mathscr{R}_0}}\right\}$$

and $\mu < \frac{2\beta_e\Lambda}{\mu\mathscr{R}_0}$, $\xi_2 + 2\frac{\Lambda}{\mu\mathscr{R}_0} < w + \mu$ and $2\beta_v\Lambda < \mathscr{R}_0\sigma$.

Similarly, the disease-free equilibrium of the discrete system (2) is (locally) asymptotically stable if

$$h < min\left\{\frac{2}{\xi_1 + \xi_2 + \sigma}, \frac{2}{\alpha + w + \gamma + \mu}, \frac{2}{\mu}, \frac{2}{\alpha + \mu + (\beta_i + \beta_v - \beta_e)\frac{\Lambda}{\mu}}\right\}$$

and $\alpha < w + \gamma + \mu$, $\xi_1 + \xi_2 < \sigma$ and $(\beta_e + \beta_i + \beta_v)\frac{\Lambda}{\mu} < \alpha + \mu$, or if

$$h < min\left\{\frac{2}{\sigma + \beta_v\frac{\Lambda}{\mu}}, \frac{2}{\alpha + w + \gamma + \mu + \xi_1 + \beta_i\frac{\Lambda}{\mu}}, \frac{2}{\mu}, \frac{2}{2\alpha + \mu + \xi_1 + \beta_e\frac{\Lambda}{\mu}}\right\}$$

and $\beta_i\frac{\Lambda}{\mu} + \xi_2 < w + \gamma + \mu$, $\beta_v\frac{\Lambda}{\mu} < \sigma$ and $\mu < \beta_e\frac{\Lambda}{\mu} + \xi_1$. If $\mathscr{R}_0 > 1$ then the discrete system (2) is unstable at the disease-free equilibrium for all h.

Note that the conditions for the parameters have no biological meaning and the given conditions only states local stability. It is also possible to use so-called weighted Gershgorin sets to get other sufficient conditions[7]. To specify conditions for the global stability, one must use different methods, for example Lyapunov functions[8].

3 Discussion,Future Directions

By constructing epidemiological models (with the help of mathematics) we can understand the dynamics and their qualitative characteristics of the different infectious diseases. We can also make predictions and test the impact of different control strategies. Within the framework of Math Project II., I considered a COVID-19 model[1], which incorporates an environmental reservoir as a compartment with its own dynamics. I gave sufficient conditions on the preservation of a positively invariant (biologically feasible) region for its explicit Euler discretized system. I also gave sufficient conditions for the step-sizes and the parameters for the stability of the two equilibria. For future directions, it would be interesting to check the preservation of these properties for other numerical methods, for example the implicit Euler method. Another possibility is to check these with other incidence rates, for example the saturated incidence rate, where $\beta_e, \beta_i, \beta_v$ is not constant but a function of E, I, V: $\beta_E(E) = \frac{\beta_{E0}}{1+cE}$, $\beta_I(I) = \frac{\beta_{I0}}{1+cI}, \beta_V(V) = \frac{\beta_{V0}}{1+cV}$, respectively. In the written report of last semesters Math Project I., the integration of the environmental reservoir into different patch models was given as a possible future direction. While I was able to incorporate this additional compartment to two different patch models, I was not able to show any common properties of these models. Deriving some properties for these models can also be considered as a possible future direction.

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