Numerical modelling of disease propagation

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Math project II. presentation

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2020, Yang and Wang proposed the following compartmental model[1]:

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \beta_E SE - \beta_I SI - \beta_V SV - \mu S \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \beta_E SE + \beta_I SI + \beta_V SV - (\alpha + \mu)E \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \alpha E - (w + \gamma + \mu)I \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I - \mu R \\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \xi_1 E + \xi_2 I - \sigma V
\end{aligned} \tag{1}
$$

Parameters

Λ	Population influx	ξ_1	Rate of the exposed individuals contributing the virus to the environment
μ	Natural death rate		
w	Disease induced death rate	ξ_2	Rate of the infected individuals contributing the virus to the environment
$1/\alpha$	Mean incubation period		
γ	Recovery rate	σ	Rate of (natural and artificial) removal of the virus from the environment
β_I	Transmission rate by infected individual		
β_E	Transmission rate by exposed individual		
β_V	Transmission rate by the environmental reservoir		

Explicit Euler discretization

$$
\begin{aligned}
s_{n+1} &= s_n + h(\Lambda - \beta_E s_n e_n - \beta_I s_n i_n - \beta_V s_n v_n - \mu s_n) \\
e_{n+1} &= e_n + h(\beta_E s_n e_n + \beta_I s_n i_n + \beta_V s_n v_n - (\alpha + \mu) e_n) \\
i_{n+1} &= i_n + h(\alpha e_n - (w + \gamma + \mu) i_n) \\
r_{n+1} &= r_n + h(\gamma i_n - \mu r_n) \\
v_{n+1} &= v_n + h(\xi_1 e_n + \xi_2 i_n - \sigma v_n)
\end{aligned} \tag{2}
$$

Property preservation of the discrete model:

- (2) Can be interpreted as a discrete dynamical system

Positively invariant region

For the continuous case

$$\Omega = \left\{ (S, E, I, R, V) \in \mathbb{R}^5_+ : S + E + I + R \leq \frac{\Lambda}{\mu}, 0 \leq V \leq \frac{(\xi_1 + \xi_2)\Lambda}{\mu\sigma} \right\} \quad (3)$$

is positively invariant region[1].

Theorem

The discretized system (2) *is positively invariant in* Ω *if* $h \leq min\{\frac{1}{\mu + (\beta_e + \beta_i + \beta_v)\frac{\Lambda}{\mu}}, \frac{1}{\alpha + \mu}, \frac{1}{w + \gamma + \mu}\}$ *and* $h < \frac{1}{\sigma}$.

There is a necessary condition

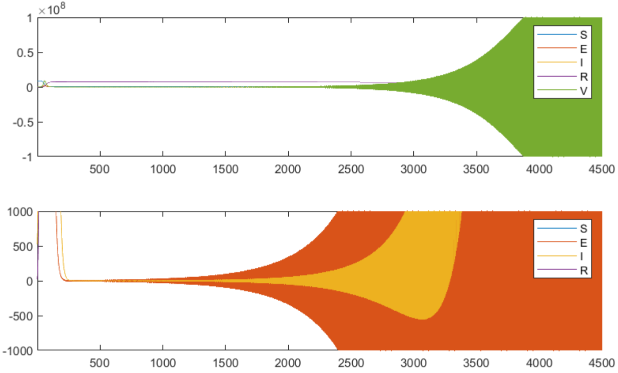

Figure: Negativity of the Explicit Euler method when $h = 2$. The lower figure is the same as the upper one, but the variable V is not shown, and it is also zoomed in.

Equilibrium points

- Discrete system: $\mathbf{x}_{n+1} = g(\mathbf{x}_n)$
- $\mathbf{x}^*$ equlibrium: $\mathbf{x}^* = g(\mathbf{x}^*)$
- The discrete model has the same equilibrium points as the continous model.

Disease-free equilibrium (DFE)

- no infections in the population
- $\mathscr{E}_0 = (S_0, E_0, I_0, R_0, V_0) = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$

Endemic equilibrium:

$$s^* = \frac{\alpha + \mu}{\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_V}$$

$$e^* = \frac{\Lambda}{\alpha + \mu} - \frac{\mu}{\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_V}$$

$$i^* = \frac{\Lambda\alpha}{w_1(\alpha + \mu)} - \frac{\alpha\mu}{w_1(\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_V)}$$

$$r^* = \frac{\gamma\alpha\Lambda}{\mu w_1(\alpha + \mu)} - \frac{\gamma\alpha}{w_1(\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_V)}$$

$$v^* = \frac{}{\alpha + \mu} - \frac{c\mu}{\beta_E + \frac{\alpha}{w_1}\beta_I + c\beta_v}$$

where $c = \frac{w_1\xi_1 + \xi_2\alpha}{\sigma w_1}$, $w_1 = \gamma + \mu + w$

Equilibria - continuous case

The continous system exhibits forward bifurcation:
In the case of $\mathscr{R}_0 < 1$, the boundary equilibrium is globally asymptotically stable on Ω and the endemic equilibrium is unstable, while in the case $\mathscr{R}_0 > 1$ the endemic equilibrium is globally asymptotically stable on Ω and the disease-free equilibrium is unstable.[1]

$$\mathcal{R}_0 = \frac{\beta_E S_0}{\alpha + \mu} + \frac{\beta_I S_0 \alpha}{(\alpha + \mu)(w + \gamma + \mu)} + \left(\frac{\beta_V S_0 \xi_1}{(\alpha + \mu)\sigma} + \frac{\beta_V S_0 \alpha \xi_2}{(\alpha + \mu)(w + \gamma + \mu)\sigma} \right)$$
$$=: \mathcal{R}_E + \mathcal{R}_I + \mathcal{R}_V$$

Number of secondary infections produced by an initially exposed individual in a completely susceptible population. (Conditions same as having the real part of the eigenvalues of the Jacobian be negative)

Equilibria - discrete case

By Gerschorin disks for the eigenvalues of the Jacobian at the equilibria. Let $A = [a_{ij}]_{i,j=1}^{N}$ be a real quadratic matrix (with $a_{ii} \in (-1,1)$), then to have all of its eigenvalues in the unit disk, we must have $a_{ii} - R^i(A) > -1$ and $a_{ii} + R^i(A) < 1$ for all of its rows, where $R^i(A)$ is the i-th deleted absolute row sum of A.

One example

One sufficient condition for the asymptotic stability of the endemic equilibria:

$$h < \min\left\{\frac{2}{\mu}, \frac{2}{\mu(2\mathscr{R}_0 - 1)}, \frac{2}{2\alpha + \mu + \xi_1}, \frac{2}{w + 2\lambda + \mu + \xi_2 + 2\beta_i \frac{\Lambda}{\mu\mathscr{R}_0}}, \frac{2}{\sigma + 2\beta_v \frac{\Lambda}{\mu\mathscr{R}_0}}\right\}$$

and $\mu < \dfrac{2\beta_e\Lambda}{\mu\mathscr{R}_0}$, $\xi_2 + 2\dfrac{\Lambda}{\mu\mathscr{R}_0} < w + \mu$ and $2\beta_v\Lambda < \mathscr{R}_0\sigma$.

If $\mathscr{R}_0 < 1$ then the discrete system (2) is unstable at the endemic equilibrium for all h.

- These conditions are sufficient but not neccesary and guarantees local and not global stability.

Sufficient and necessary conditions for $\mathcal{E}_0$

The characteristic polynomial of the Jacobian at $\mathscr{E}_0$ is:

$$F^*(\lambda) = (1 - h\mu - \lambda)^2 F(\lambda) = (1 - h\mu - \lambda)^2(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3)$$

where

$$\begin{aligned} b_1 &= -3 - hs_1 \\ b_2 &= 3 + 2hs_1 + h^2 s_2 \\ b_3 &= -1 - hs_1 - h^2 s_2 + h^3 s_3 \end{aligned}$$

and

$$\begin{aligned} s_1 =& - \sigma - \alpha - 2\mu - w - \gamma + \beta_e S_0 \\ s_2 =& \sigma(\alpha + \mu) + \sigma(w + \gamma + \mu) + (\alpha + \mu)(w + \gamma + \mu) - \\ & - \beta_i S_0 \alpha - \beta_v \xi_1 S_0 - \beta_e S_0 (w + \gamma + \mu) - \sigma \beta_e S_0 \\ s_3 =& \sigma(\alpha + \mu)(w + \gamma + \mu) - \sigma \beta_e S_0 (w + \gamma + \mu) - \\ & - \beta_v S_0 \alpha \xi_2 - \beta_i S_0 \alpha \sigma - \beta_v S_0 \xi_1 (w + \gamma + \mu) \end{aligned}$$

Sufficient and necessary conditions for $\mathcal{E}_0$

Schur-Cohn criterium[2]:

(i.) $F(1) > 0$

(ii.) $-F(-1) > 0$

(iii.) $|1 - b_3^2| > |b_2 - b_1 b_3|$

Future directions

- Implicit Euler scheme
- Saturated incidence rates:
  $\beta_E(E) = \dfrac{\beta_{E0}}{1 + cE}, \ \beta_I(I) = \dfrac{\beta_{I0}}{1 + cI}, \ \beta_V(V) = \dfrac{\beta_{V0}}{1 + cV}$

References I

[1]–Yang, Chayu, and Jin Wang. "A mathematical model for the novel coronavirus epidemic in Wuhan, China." Mathematical biosciences and engineering: MBE 17.3 (2020): 2708.
[2]–S. Elaydi. "An Introduction to Difference Equations." Springer-Verlag, 2005, isbn: 978-1-4419-2001-0.