Presentation of Individual Project II.

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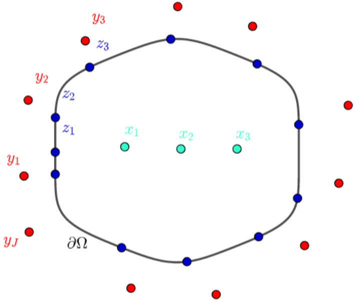
19 May 2022

Importance of PDE's

- Describe the evolution of continuous systems
- Numerous applications in STEM, examples:
  - Transport equation: $\partial_t u + \langle v, \operatorname{grad} u \rangle = 0$, ($v \in \mathbb{R}^n$ fixed)
  - Heat (diffusion) equation: $\partial_t u - \alpha \Delta u = 0$, ($\alpha \in \mathbb{R}$ fixed)
  - Wave equation: $\partial_t^2 u - c^2 \Delta u = 0$, ($c \in (0, +\infty)$ fixed)
  - Laplace and Poisson's equations: $\Delta u = f$, ($f \in L^2(\Omega)$ fixed)
  - and many more...

Neural Networks $\Longrightarrow$ PDE solution

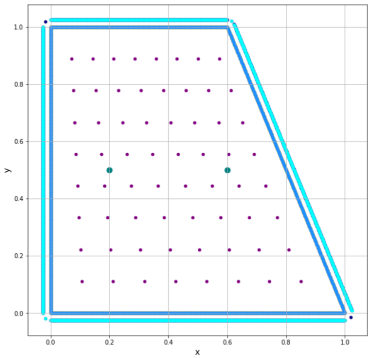
[2] shows working mechanism to teach a NN to estimate the solution of Laplace's equation ($\Delta u = 0$ with a non-homogenous Dirichlet-type boundary condition) in a nontrivial $\Omega \subseteq \mathbb{R}^2$:



Now

- We try to modify the previous method to give a solution for Poisson's equation ($\Delta u = f$ with a Dirichlet boundary condition)
- Obviously we need information from inside Ω
- Fundamental functions $\rightarrow$ Radial functions

The domain and explanations



- Cyan: $W = \{w_l\}_{l=0}^{L-1} \subset \Omega$ a number of exterior points, which serve as a center of the Green functions
- Blue: $Z = \{z_k\}_{k=0}^{K-1} \subset \partial\Omega$ boundary points,
- Purple: $Y = \{y_j\}_{j=0}^{J-1} \subset \overline{\Omega}$ be basis points of radial functions.
- Dark cyan: $X = \{x_i\}_{i=0}^{I-1} \subset \Omega$ be a set of a few interior points where the solution is to be approximated,

Goals for this semester:

- First attempt: linear regression (not really NN yet)
- Defining and verifying the correct data structures,
- Picking appropriate platform,
- Feeding data to NN
- Tweaking the number of points, radial functions used for better performance.

Data to be fed

- Belonging to the points $\mathrm{y_j}$ – for every $j \in \{0, 1, \ldots, J-1\}$ – an input-output pair:

$$\underbrace{(\Delta\Psi_{\mathrm{y_j}}(T), \Psi_{\mathrm{y_j}}(Z))}_{\in\mathbb{R}^{M+K} \text{ input}} \rightarrow \underbrace{\Psi_{\mathrm{y_j}}(X)}_{\in\mathbb{R}^{I} \text{ output}},$$

- and belonging to the points $\mathrm{w_l}$ – for every $l \in \{0, 1, \ldots, L-1\}$ – an input-output pair:

$$\underbrace{(0, G_{\mathrm{w_l}}(Z))}_{\in\mathbb{R}^{M+K} \text{ input}} \rightarrow \underbrace{G_{\mathrm{w_l}}(X)}_{\in\mathbb{R}^{I} \text{ output}}.$$

This amounts to a total number of $J + L$ input-output pairs. Now let the the test set take the form of

$$((f(T)), (g(Z))) \in \mathbb{R}^{M+K}.$$

Data to be fed: matrix representation

The input matrix takes the form

$$\begin{pmatrix}
\Delta\Psi_{y_0}(t_0) & \Delta\Psi_{y_0}(t_1) & \cdots & \Delta\Psi_{y_0}(t_{M-1}) & \Psi_{y_0}(z_0) & \Psi_{y_0}(z_1) & \cdots & \Psi_{y_0}(z_{K-1}) \\
\Delta\Psi_{y_1}(t_0) & \Delta\Psi_{y_1}(t_1) & \cdots & \Delta\Psi_{y_1}(t_{M-1}) & \Psi_{y_1}(z_0) & \Psi_{y_1}(z_1) & \cdots & \Psi_{y_1}(z_{K-1}) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\Delta\Psi_{y_{J-1}}(t_0) & \Delta\Psi_{y_{J-1}}(t_1) & \cdots & \Delta\Psi_{y_{J-1}}(t_{M-1}) & \Psi_{y_{J-1}}(z_0) & \Psi_{y_{J-1}}(z_1) & \cdots & \Psi_{y_{J-1}}(z_{K-1}) \\
0 & 0 & \cdots & 0 & G_{w_0}(z_0) & G_{w_0}(z_1) & \cdots & G_{w_0}(z_{K-1}) \\
0 & 0 & \cdots & 0 & G_{w_1}(z_0) & G_{w_1}(z_1) & \cdots & G_{w_1}(z_{K-1}) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & G_{w_{L-1}}(z_0) & G_{w_{L-1}}(z_1) & \cdots & G_{w_{L-1}}(z_{K-1})
\end{pmatrix}$$

while the output matrix looks as follows

$$\begin{pmatrix}
\Psi_{y_0}(x_0) & \Psi_{y_0}(x_1) & \cdots & \Psi_{y_0}(x_{I-1}) \\
\Psi_{y_1}(x_0) & \Psi_{y_1}(x_1) & \cdots & \Psi_{y_1}(x_{I-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\Psi_{y_{J-1}}(x_0) & \Psi_{y_{J-1}}(x_1) & \cdots & \Psi_{y_{J-1}}(x_{I-1}) \\
G_{w_0}(x_0) & G_{w_0}(x_1) & \cdots & G_{w_0}(x_{I-1}) \\
G_{w_1}(x_0) & G_{w_1}(x_1) & \cdots & G_{w_1}(x_{I-1}) \\
\vdots & \vdots & \ddots & \vdots \\
G_{w_{L-1}}(x_0) & G_{w_{L-1}}(x_1) & \cdots & G_{w_{L-1}}(x_{I-1})
\end{pmatrix}$$

Dimensions and linear regression

- $J + L$ input vectors, each of length $M + K$
- $J + L$ corresponding output vectors, each of length I
- First trial: find $L \in V := \mathcal{L}(\mathbb{R}^{M+K}, \mathbb{R}^I)$ linear map.
- dim V should be relatively close to the parameters of the NN.
- Given L, the numerical estimate $\big(\tilde{u}(\mathrm{x}_0), \tilde{u}(\mathrm{x}_1)^T, \ldots \tilde{u}(\mathrm{x}_{\mathrm{I}-1})\big)$ should be $L \cdot v$, where

$$v = \big(\Delta f(\mathrm{t}_0), f(\mathrm{t}_1), \ldots, \Delta f(\mathrm{t}_{\mathrm{M}-1}), g(\mathrm{z}_0), g(\mathrm{z}_1), \ldots, g(\mathrm{z}_{\mathrm{K}-1})\big)^T$$

Platform

- Visual Studio Code and local experiments.
- Google Colaboratory and GPU enhanced experiments.

Tweaking parameters

- One problem: I, J, K, L are not independent at the moment.
- It's reasonable to expect that $\dim V = I \cdot (M + K)$ should be about $J + L$
- Tweaking the parameters like this did not yield the expected results

Experiments

- Refer to the PDF.

Near future goals

- Making the linear model relatively efficient.
- Restructuring data: convolutional approach
- Experimenting with different NN structures
- Storing weights so that
- different f and g functions may be given as inputs

Far future

- Experimenting with techniques from image processing
- Different types of differential operators
- Time dependence
- Different Ω
- More dimensions
- ... and whatever we may imagine.

Bibliography

- Cheng, A.H.D and Hong, Y.: An overview of the method of fundamental solutions—Solvability, uniqueness, convergence, and stability, Engineering Analysis with Boundary Elements 120, pp. 118-152, 2020.
  `idemajdkelllink`
- Haffner, D. and Izsák, F.: Solving the Laplace equation by using neural networks, to appear in Annales Univ. Sci. Budapest., Sec. Math.
  `idemajdkelllink`
- Isaac Elias Lagaris, Aristidis Likas and Dimitrios I. Fotiadis: Artificial Neural Networks for Solving Ordinary and Partial Differential Equations, IEEE Transactions on Neural Networks, vol. 9, no. 5, 1998. szeptember
  https://faculty.sites.iastate.edu/hliu/files/inline-files/Anns_lagaris1998_0.pdf