# 2021/2022 First Semester Research Project

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## 1 Introduction

This semester, I've primarily studied books and articles related to algebraic geometry over $\mathbb{C}$. I've studied the Lefschetz theory of complex algebraic varieties, and the Hodge structure of Kähler manifolds and varieties.

## 2 The Lefschetz Theory of Complex Varieties

For this section, I've read Lamotke's paper [1].

A pencil in $\mathbb{CP}^N$ is the set of hyperplanes that contain a given $(N-2)$-dimensional linear subspace. The axis of the pencil is the common $(N-2)$-dimensional subspace. Note that a pencil is just a projective line $G \subset \left(\mathbb{CP}^N\right)^*$ in the dual space (hence homeomorphic to $S^2$). The Lefschetz theory of $\mathbb{C}$ varieties considers the (co)homology of a smooth (nonsingular) irreducible variety intersected with the hyperplanes of a pencil. From the homology of these intersections, we can derive strong structure theorems of the homology of the variety, using the monodromy of a fiber bundle.

Let $X$ denote the smooth irreducible variety in $\mathbb{CP}^N$. We have the following proposition:

**Proposition 2.1.**

1. *Hyperplanes that are tangent to $X$ form a closed irreducible subvariety $X^* \subset \left(\mathbb{CP}^N\right)^*$. $X^*$ is called the dual variety of $X$.*
2. *The hyperplanes that intersect $X$ transversally form the Zariski-open set $\left(\mathbb{CP}^N\right)^* \setminus X^*$.*

Instead of working with $X$, we consider the blowup of $X$ along the axis of the given pencil $G$ with axis $A$: $Y = \{(x,t) \in X \times G | x \in H_t\}$, where $H_t$ denotes the tangent hyperplane of $X$ corresponding to $t \in \left(\mathbb{CP}^N\right)^*$ ($X$ is smooth). We denote $X' := X \cap A$, $Y' := X' \times G$.

**Proposition 2.2.**

1. *$Y$ is an irreducible smooth variety.*
2. *The projection $f : Y \to G$ has $r = \operatorname{class} X$ critical values, the points $X^* \cap G$. Here $\operatorname{class} X$ is the degree of the dual variety $X^*$.*
3. *Every critical value of $f$ is nondegenerate ($f$ is Morse), whenever $G$ is generic.*

The main idea is to decompose the projective line $G$, which is homeomorphic to an $S^2$, into two closed hemispheres $D_+, D_-$ such that the critical values of $f$ lie in $\operatorname{int} D_+$, and choosing a point $b \in S^1 = D_+ \cap D_-$. We denote the intersection of $X$ with the hyperplane corresponding to $b$ by $X_b$.

**Theorem 2.3** (Lefschetz)**.** $H_q(X, X_b) = 0$ *for all* $q \leq \dim X - 1$

This is proven using the lemma that Lamotke calls the "main lemma of Lefschetz":

**Theorem 2.4** (Main Lemma)**.** $H_q(Y_+, Y_b) = 0$ *if* $q \neq \dim X$*, and* $H_q(Y_+, Y_b)$ *is free of rank* $r = \operatorname{class} X$*.*

We take a more detailed look on the homology at $n := \dim X$. From the long exact sequence of relative homology, we have a connecting homomorphism $\partial_* : H_n(Y_+, Y_b) \to H_{n-1}(Y_b)$. Its image is called the module of vanishing cycles, denoted $V$; and agrees with the kernel of $i_* : H_{n-1}(X_b) \to H_{n-1}(X)$. In a similar fashion, we define the module $I^*$, the module of invariant cocylces as the image of $i^* : H^{n-1}(X) \to H^{n-1}(X_b)$. The module of invariant cycles is then $I$, the Poincaré-dual of $I^*$.

By using properties of Poincaré-duality, we have the following theorem.

**Proposition 2.5.**

1. $\operatorname{rank} H_{n-1}(X_b) = \operatorname{rank} V + \operatorname{rank} H_{n-1}(X)$
2. $\operatorname{rank} I = \operatorname{rank} H_{n+1}(X) = \operatorname{rank} H_{n-1}(X)$.
3. $\operatorname{rank} I + \operatorname{rank} V = \operatorname{rank} H_{n-1}(X_b)$.

In fact, if field coefficients are chosen, something much stronger holds:

**Theorem 2.6** (Hard Lefschetz)**.** $H_{n-1}(X_b) = I \oplus V$*.*

Note that similarly to the Hard Lefschetz theorem, in the following section, field coefficients are chosen. The proof of the Hard Lefschetz theorem is not included in the article. In Arapura's book [2], a cohomological version is proved with completely different methods, namely using Hodge theory. For the details see section 3.

We take the sequence $X' \subset X_b \subset X$, and extend it into the following:

$$0 = X_{n+1} \subset X_n \subset \ldots \subset X_3 \subset X_2 = X' \subset X_1 = X_b \subset X_0 = X$$

by making each $X_i$ to be a generic hyperplane section of $X_{i-1}$, hence $\dim X_i = n - i$. (Note that during this study, we always mean dimensions over $\mathbb{C}$). Let $u \in H^2(X)$ denote the Poincaré dual of the fundamental class $[X_b] \in H_{2n-2}(X)$. We have that the Poincaré-duals $[X_q]^* \in H^{2q}(X)$ agree with $u^q$. Lamotke proves in his paper, that the Hard Lefschetz theorem is equivalent to the following statement:

**Theorem 2.7** (Hard Lefschetz 2)**.** *For all* $q = 1, \ldots n$*, we have*

$$H_{n+q}(X) \simeq H_{n-q}(X),\ x \mapsto u^q \cap x$$

Note that by Poincaré duality, in case of field coefficients we have isomorphisms $H_{n+q}(X) \simeq H_{n-q}(X)$, the statement of the above theorem is about different isomorphisms, which are induced by the cap product with the fundamental class of $X_b$. From the above form of the Hard Lefschetz, we derive the equally important statement:

**Theorem 2.8** (Hard Lefschetz 3 - Primitive Decomposition). *$\forall x \in H_{n+q}$ $\exists! x_0, x_1, \ldots$ s.t. $x = x_0 + u \cap x_1 + u^2 \cap x_2 + \ldots$, and $\forall x \in H_{n-q}$ $\exists! x_0, x_1, \ldots$ s.t. $x = u^q \cap x_0 + u^{q-1} \cap x_1 + \ldots$, where the above $x_i$ are all primitive elements, i.e $u^{q+1} \cap x = 0$ (note that $q+1$ is the smallest such index $j$ for which a nonzero $x$ can have the property that $u^j \cap x = 0$).*

The last form of the Hard Lefschetz theorem considers the Lie-algebra $\mathrm{sl}_2$ of $2 \times 2$ matrices with trace 0. $\mathrm{sl}_2$ is 3-dimensional, with the following basis elements:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

satisfying the Lie-bracket relations:

$$[eh] = -2e, \quad [fh] = 2f, \quad [ef] = h.$$

**Theorem 2.9** (Hard Lefschetz 4 - $\mathrm{sl}_2$-module). *$H_*(X)$ is an $\mathrm{sl}_2$-module.*

Here

$$f : H_j(X) \to H_{j-2}(X), \quad x \mapsto u \cap x$$
$$h : H_j(X) \to H_j(X), \quad x \mapsto (j-n)x.$$

The definition of $e$ requires the primitive deomposition of the cohomology ring. We define $e$ to be $e(u^r \cap x) = r(q-r+1)u^{r-1} \cap x$. Then $f, h, e$ are endomorphisms of $H_*(X)$, that satisfy the commutator relations of the $e, f, h$ basis elements of $\mathrm{sl}_2$. Hence, we have a representation $\mathrm{sl}_2 \to gl(H_*(X))$, which turns $H_*(X)$ indeed into an $\mathrm{sl}_2$-module.

For the last part of this section, we consider some methods used to prove the main lemma (2.4).

When we take the space $G^* = G \setminus \{t_1, \ldots t_r\}$, where $r$ is the class of $X$, the $t_i$ are the critical values of the projection $f : Y \to G$, and choose a regular value $b \in G$; the fundamental group $\pi_1(G^*, b)$ acts on the homology of $Y_b$ (because $Y^* = f^{-1}(G^*)$ is a locally trivial fiber bundle over $G^*$ with generic fibers $Y_b = X_b$). If $w_i$ is the loop that goes from $b$ to "almost" $t_i$, circles $t_i$ once counterclockwise and returns to $b$, then $\pi_1(G^*, b)$ is generated by the homotopy classes $[w_1], [w_2], \ldots [w_r]$. Each $w_i$ consists of a line $l_i$ and a circle $S_i$ that is the boundary of a disk $D_i$. Let $L$ denote the preimage $f^{-1}(\bigcup l_i)$, $K = L \cup f^{-1}(\bigcup D_i)$. The $l_i$ can be chosen disjoint, so that $K$ is a strong deformation retract of $Y_+$, and $L$ is a deformation retract of $Y_b$. We will also use the notation $T_i = f^{-1}(D_i)$ and $F_i = f^{-1}(a_i)$, where $a_i$ is the point in which $l_i$ reaches $D_i$. Since $f$ is Morse, we can choose coordinates such that $f(z) = t_i + z_1^2 + \ldots z_n^2$, in a suitably small neighbourhood of a critical point

$x_i$, where $f(x_i) = t_i$. This local neighbourhood is denoted $(T, F)$ in accordance with the notation $T_i, F_i$.

The orientation of the real $n$-disk $\Delta = \{z \in T | \text{all } z_i \text{ are real}\}$ freely generates $H_n(T, F)$, for $q \neq n$, $H_q(T, F) = 0$. We have elements $[\Delta_i] \in H_n(Y_+, Y_b)$, induced by the inclusions, and these generate $H_n(Y_+, Y_b)$ freely. The connecting homomorphism gives us

$$\delta_i = \partial_* \Delta_i \in H_{n-1}(Y_b).$$

Such elements are called vanishing cycles. The name is motivated by the fact, that the vanishsing cycles are exactly the ones that vanish at the critical point, when moved among its thimble $\Delta_i$.

Upon investigation, we find that

**Proposition 2.10.** *The self-intersection number $(\delta_i, \delta_i)$ is 0 for $n$ even, and $(-1)^{(n-1)/2} \cdot 2$ for $n$ odd.*

This gives us the Picard-Lefschetz formula:

**Theorem 2.11.** *If $q \neq n-1$ then $\pi_1(G^*, b)$ acts trivially on $H_q(Y_b)$. For $q = n-1$, the elementary path $w_i$ acts by $(w_i)_*(x) = x + (-1)^{(n-1)/2}(x, \delta_i)\delta_i$.*

The proof of the Picard-Lefschetz formula requires topological examination of the monodromy, mostly omitted here.

The module of invariant cocylces contains exactly the elements of $H_{n-1}(Y_b)$ that are invariant under the action of $\pi_1(G^*, b)$. One further important result is the following:

**Theorem 2.12** (Monodromy Theorem)**.** *If we have coefficients in a field, then TFAE:*

1. *The Hard Lefschetz theorem.*
2. $V = 0$
3. *$H_{n-1}(Y_b)$ is a semisimple $\pi_1(G^*, b)$-module.*

# 3 Hodge Theory

Dureing the second half of the semester, I studied the Hodge theory of Riemannian manifolds, and then Kähler manifolds.

For an orientable Riemannian (real, smooth) manifold, the inner product on the top exterior power $\mathcal{E}_X^n$ gives a tensor $\det(g) \in \Gamma(X, \mathcal{E}_X^n \otimes \mathcal{E}_X^n)$, where $g$ denotes the inner product defined by the metric. This in turn gives us a volume form $dvol$, which is well-defined because the manifold can be oriented. The Hodge-star operator is defined by

$$\alpha \wedge *\beta = (\alpha, \beta) dvol,$$

it is $\mathcal{C}^\infty(X)$-linear. The main use of the Hodge-star is the following identity

$$\langle \alpha, \beta \rangle = \int_X (\alpha, \beta) dvol = \int_X \alpha \wedge *\beta.$$

**Theorem 3.1** (The Hodge Theorem)**.** *Every de Rham cohomology class has a unique representative that minimalizes the norm. This is called the harmonic representative.*

If $d^*$ is the adjoint of $d$ with respect to the above defined inner product, then a form is harmonic iff $d^*\alpha = d\alpha = 0$. By defining the Hodge Laplacian $\Delta = d^*d + dd^*$, we get the further characterization: $\alpha$ is harmonic iff $\Delta\alpha = 0$.

In the book of Arapura, certain application of the Hodge theorem and Hodge Laplacian are investigated. For example a version of the Poincaré-duality (formulated with perfect pairings) is proven. I did study these sections, together with the proof of the Hodge theorem (which uses the so called heat equation of the Riemannian manifold), but in order to keep the presentation short, I omit them.

When examining the structure of $\mathbb{C}$ manifolds, we divide the sheaf of differential forms into subsheafs. Given an $n$-dimensional $\mathbb{C}$-manifold with $\mathcal{O}_X$ sheaf of holomorphic functions, $\mathcal{E}_X^k$ is the sheaf of complex-valued $\mathcal{C}^\infty$ forms. We denote by $\Omega_X^p$ the sheaf of holomorphic $p$-forms, which is a subsheaf of $\mathcal{E}_X^p$ stable under multiplication by $\mathcal{O}_X$. We get further structure on the manifold by using the following definition:

**Definition 3.2.** $\mathcal{E}^{(p,0)}$ is the $\mathcal{C}^\infty$-submodule of $\mathcal{E}_X^p$ generated by $\Omega_X^p$. $\mathcal{E}^{(0,p)} = \overline{\mathcal{E}^{(p,0)}}$, and $\mathcal{E}^{(p,q)} = \mathcal{E}^{(p,0)} \wedge \mathcal{E}^{(0,q)}$.

**Theorem 3.3** (Dolbeault's theorem)**.** *For any complex manifold $X$,*

1. $0 \to \Omega_X^p \to \mathcal{E}^{(p,0)} \xrightarrow{\overline{\partial}} \mathcal{E}^{(p,1)} \xrightarrow{\overline{\partial}} \dots$ *is a soft resolution (i.e a resolution in which each element is a soft sheaf).*

2. $$H^q(X, \Omega_X^p) \simeq \frac{\ker\left(\mathcal{E}^{(p,q)} \to \mathcal{E}^{(p,q+1)}\right)}{\operatorname{im}\left(\mathcal{E}^{(p,q-1)} \to \mathcal{E}^{(p,q)}\right)}$$

Similarly to the Riemannian case, several operators are introduced: $\partial^*, \Delta_\partial, \Delta_{\overline{\partial}}$. The various relations of these operators are key in the methods of Hodge theory. For compactness sake, we omit these theorems.

We wish to extend (in some form) the Hodge theorem for complex manifold. As it turns out, this can only be done for a special subclass of complex manifolds, the so called Kähler manifolds. We call a Riemannian metric on a $\mathbb{C}$-manifold Hermitian, if the multiplication by $\sqrt{-1}$ is orthogonal. Equivalently, it is a hermitian inner product defined on the tangent spaces, that varies in a $\mathcal{C}^\infty$ fashion. If $z_i = x_i + \sqrt{-1}y_i$ are local analytic coordinates, for a Hermitian metric $H$ we have that $H = \sum h_{ij} dz_i \otimes \overline{dz_j}$, where $(h_{ij})$ is a positive definite Hermitian matrix.

**Definition 3.4.** A $\mathbb{C}$-manifold is Kähler, if it admits a Hermitian metric that is locally Euclidean up to second order, i. e. if for any point $p \in X$ there exist analytic local coordinates $z_1, \dots, z_n$ with $z_i = 0$ at p, such that

$$h_{ij} \equiv \delta_{ij} \mod (x_1, y_1, \dots, x_n, y_n)^2.$$

It is important that any smooth complex projective variety is Kähler with the Fubini-Study metric. By the Kodaira embedding theorem and Chow's theorem we in fact have that a compact complex manifold is a nonsingular projective algebraic variety iff it has a Kähler metric with rational Kähler class. Related to the Käher metric we have a Kähler form $\omega$. For $H = \sum h_{ij} dz_i \otimes \overline{dz_j}$. The image of $H$ in $\mathcal{E}_X^{(1,1)}$ is the Kähler metric $\omega$; it is real i.e. $\overline{\omega} = \omega$.

We can extend the Hodge star to $\mathcal{E}_X^*$, and define $\overline{*}(\alpha) = \overline{*\alpha}$, to get two Hodge star operators on our complex manifold. We define additional operators on $X$:

- $\Delta_\partial = \partial^*\partial + \partial\partial^*$ with bidegree $(0,0)$
- $\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$ with bidegree $(0,0)$
- $L = \omega\wedge$ with bidegree $(1,1)$
- $\Lambda = - * L *$, with bidegree $(-1,-1)$.

Using these operators, we obtain the following theorem:

**Theorem 3.5.** *Suppose that $X$ is a compact Kähler manifold.*

- $H^q(X, \Omega_X^p)$ *is isomorphic to the space of harmonic* $(p,q)$*-forms.*
- *As a corollary,* $H^p(X, \Omega_X^q) \simeq H^{n-p}(X, \Omega_X^{n-q})$

The main theorem of Hodge theory is the following:

**Theorem 3.6** (Hodge decomposition)**.** *If $X$ is a compact Kähler manifold, then*

- *A form $a$ is harmonic iff its* $(p,q)$*-components are.*
- $H^i(X, \mathbb{C}) \simeq \bigoplus_{p+q=i} H^q(X, \Omega_X^p)$
- *Complex conjugation induces an* $\mathbb{R}$*-linear isomorphisms between* $(p,q)$ *and* $(q,p)$ *forms. Therefore* $H^q(X, \Omega_X^p) \simeq H^p(X, \Omega_X^q)$.

Similarly to Betti numbers, we have Hodge numbers $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$, these are finite by the Hodge decomposition. We also obtain a Künneth-like formula on the cohomology of the sheaves $\Omega_X^p$.

I have also looked at the proof of the Hard Lefschetz theorem using Hodge theory. This uses the above-defined operators to explicitly give the $\mathrm{sl}_2$-representation of the cohomology ring.

# 4 Notes

During this semester, I have also studied some properties of algebraic surfaces, divisors (starting with Riemann surfaces, and generalizing to schemes), the related exponential exact sequence of sheaves, and (very briefly) algebraic cycles and their relation to Hodge theory, pure Hodge structure and Hodge filtration (again, briefly). As these were not the main goal, and to keep things short, I do not go into details regarding these topics.

## References


[1] Klaus Lamotke. The topology of complex projective varieties after s. lefschetz. *Topology*, 20(1):15–51, 1981.

[2] Donu Arapura. *Algebraic geometry over the complex numbers.* Springer Science & Business Media, 2012.