Modelling project report

Cost-optimal, condition-based maintenance

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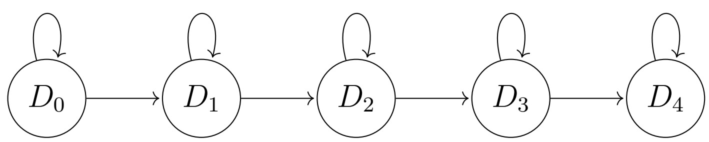
Introduction

In order to prevent catastrophic failure of a system and minimize the cost of repairing it, maintenance programs are arranged for systems which are subject to deterioration. In many cases, it is not known if an object is deteriorating or not without carrying out an inspection. Inspections are then organized aiming for early detection of deterioration [5]. However, performing a large number of inspections could be costly and unnecessary.
In this report, we present a continuous-time Markov chain as a simplified probabilistic model to deal with accumulated damage that can be described by a discrete number of states. We assume that we have five states of deterioration and we aim to provide a cost-optimal maintenance strategy. The model can be used in many application, e.g. disease progressions such as cancer, insurance claims and cost-optimal inspection policies.

Model Setup

Consider a continuous-time Markov chain $(X_t)_{t\geq 0}$ in the a space I consisting of five states D_0, D_1, D_2, D_3 and D_4 describing the level of damage. Suppose that the propagation between the states is governed by an exponential waiting time in each state. Suppose that the progression starts from the damage-free state D_0, then moves into states D_1, D_2, D_3 and last state D_4 is an absorbent state. We choose to use five states to mimic cancer progression models, where D_0 can be thought of as the disease-free state and $D_1, \ldots, D_4$ are the cancer stages at diagnosis. The aim of screening is to detect the disease as early as possible.
Let Y_i be the waiting time at state D_i before deteriorating to the next state, and suppose that $Y_i \sim exp(\lambda_i)$ for $i = 0, 1, 2, 3$.



Furthermore, suppose that the inspections may have a false positive rate, that is the probability of detecting deterioration given that the item is in D_0. Let us define the sensitivity of an inspection as the probability of detecting a deterioration given that the item is deteriorating. Suppose for now that we have a perfect sensitivity.
The aim is to derive the cost-optimal periodic inspection strategy. For that reason, let us define $\tau_0 = 0$ and $\tau_i = \tau_1 + (i-1)\Delta$ as the age at the i^{th} inspection, where Δ is the inter-inspection time. The expected total cost in this setup is divided into three parts: the expected cost of repair, the expected cost of inspections, and the expected cost associated with identifying false positives. In order to calculate the expected cost of repair, we need to derive the distribution of X_{τ_j} at each inspection τ_j and K is the total number of inspections in an observation period of length T, that is given by $K = \left\lceil \frac{T - \tau_1}{\Delta} \right\rceil$. In order to determine the distribution of X_{τ_j}, we need the density of the convolution of waiting times Y_i. For $\lambda_i \neq \lambda_j \; \forall i, j$, the density is straightforward to compute [2] and in the general case where some parameters λ_i are identical, the density can be computed

using the method proposed by Akkouchi [1]. Assuming that $\lambda_i \neq \lambda_j \; \forall i, j$, we have: [2]

$$f_{Y_0+Y_1+\cdots+Y_i}(y) = \left[\prod_{k=0}^{i} \lambda_k\right] \sum_{j=0}^{i} \frac{e^{-\lambda_j y}}{\prod\limits_{\substack{k \neq j \\ k=0}}^{i} (\lambda_k - \lambda_j)}, \quad y > 0, \;\; i \geq 1.$$

Then the probability of being in D_1 at the first inspection is given by items who moved from stage D_0 to D_1 before τ_1 and stayed there till τ_1. Therefore:

$$\begin{aligned} P(X_{\tau_1} = D_1) &= \int_0^{\tau_1} P(Y_1 > \tau_1 - s | Y_0 = s) \cdot f_{Y_0}(s) ds \\ &= \int_0^{\tau_1} \left(\int_{\tau_1 - s}^{\infty} \lambda_1 e^{-\lambda_1 y} dy \right) \lambda_0 e^{-\lambda_0 s} ds \\ &= e^{-\lambda_1 \tau_1} \lambda_0 \left[\frac{e^{(\lambda_1 - \lambda_0)\tau_1} - 1}{\lambda_1 - \lambda_0} \right]. \end{aligned}$$

And in general, the the distribution of X_{τ_j} is calculated.

Suppose that the cost of repair is an increasing function $C : I \to \mathbb{R}^+$ such that $C(i)$ is the cost of repair of an item in state D_i, then the expected total cost of repair is

$$E(CR(\tau_1, \Delta)) = \sum_{j=1}^{K} \sum_{i=0}^{4} C(i) \cdot P(X_{\tau_j} = D_i).$$

Assuming a constant false positive rate α, the expected cost associated with identifying false positives is:

$$E(CFP(\tau_1, \Delta)) = c_s \cdot \alpha \cdot \sum_{j=1}^{K} e^{-\lambda_0(\tau_j - \tau_{j-1})},$$

where c_s is the cost of identifying a false positive item.
Suppose that items participate in all inspections, the expected cost of inspection is

$$E(CI(\tau_1, \Delta)) = K \cdot r,$$

where r is the cost of a single inspection. Hence, the expected total cost is given by:

$$E(TC(\tau_1, \Delta)) = E(CR(\tau_1, \Delta)) + E(CFP(\tau_1, \Delta)) + E(CI(\tau_1, \Delta)) + E(CU),$$

where $E(CU)$ is the cost of undetected deterioration. This can be calculated after incorporating a failure state into the model.

Aims

Since it is not possible to get a closed form for the optimal τ_1 and Δ, we are currently working on implementing the model in the statistical software R. The nonlinear minimization of the expected cost will be carried out using the function *nlm*. [3] [4]
In the future, we aim to release the perfect sensitivity assumption, that means that the probability of not detecting a deterioration will no longer be zero. The second aim is to allow imperfect repair, that means that state D_4 will no longer be an absorbent state and the repair no longer moves the item back to D_0, instead we may have positive transitive probability from D_3 to D_2 or D_1 and so on. These two generalizations will render the model applicable in many scenarios.

Bibliography

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