Efficient Generator System for Stochastic Integrals with Respect to Isonormal Processes

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1 Introduction

There is now a great deal of interest in parameter estimation of Ornstein-Uhlenbeck processes with fractional driving noise in finance because of its modelling capability (see e.g. [1, 3]). That is why we investigate prediction of the unknown parameters of certain transformed fractional Ornstein-Uhlenbeck processes, e.g. Stochastic Correlation Processes, with neural networks in the hope that we will obtain more accurate estimators than the ones using classical statistic methods, without any assumptions on the Hurst exponent. As it will be precisely introduced in the second section, the published parameter estimators based on classical statistics possess such large asymptotic variance, which reduces their applicability, especially for the parameters chosen close to zero. The importance of an efficient data generator system for each analysed process rises high in this case, since if huge and complex neural network structures are applied in the learning procedure, then one needs a big amount of data for a good performance.

If one aims at simulating stochastic integrals with respect to fractional Wiener process, first of all an efficient fractal noise generator has to be developed. Several exact methods have been published about simulating fractional Wiener processes, e.g. the Cholesky and the Hosking method [6], which will be precisely determined in the third section. It will be shown that the circulant embedding based algorithms, such as [9, 12, 13], perform the most efficiently among the exact methods according to the main advantage derived from their complexity of order $\mathcal{O}(N\log N)$, where N denotes the number of grid points used in generation. All of the mentioned methods focus on calculating the square root of the covariance matrix, in the least computionally demanding way, to obtain a realization of fractional Wiener increments. Because of its efficiency, the idea developed by Davies and Harte [7] will be generalised for simulating isonormal integrals.

An efficient circulant matrix embedding method based isonormal integral simulator will be introduced, which can be applied for generating fractional Ornstein-Uhlenbeck processes with zero initial value. A Python library has been implemented for isonormal integral simulation and several comparisons as well as tests has been evaluated for different parameter sets. Even the generalised task, i.e. considering arbitrary isonormal processes instead of Wiener integrals, needs eight-times less execution time for simulating 10000 fractional Ornstein-Uhlenbeck sequences over 1500 grid-points then Kroese's method [13], which is four-times faster then the standard Davies-Harte method [7].

2 Parameter Estimation of Fractional Ornstein-Uhlenbeck Processes

As there are the many parameter configuration options, the Ornstein-Uhlenbeck processes have several use cases in mathematical finance. If one aims at adjusting the long- and short-term dependencies of the driving noise, then fractional noise driven processes could be taken into account. One of the applications of fractional Ornstein-Uhlenbeck processes is modelling the correlation between two stock prices with stochastic correlation processes, which can be obtained as a transformed fractional Ornstein-Uhlenbeck process, e.g. by applying hyperbolic tangent function. That is why accurate estimators of stochastic correlation processes

are getting more and more important in finance. This work is a part of the research about the parameter estimation of stochastic processes with deep learning methods.

Let me consider the *fractional Ornstein-Uhlenbeck process* as the pathwise unique solution of the following stochastic differential equation with $\xi_0 \in \mathbb{R}$ initial value

$$d\xi_t = -\alpha\xi_t dt + \sigma dW_t^H, \tag{1}$$

where the σ diffusion parameter and the α drift parameter have to be positive real numbers, W_t^H with $H \in [0,1]$ Hurst exponent is a fractional Wiener process in its natural filtration, i.e. $\mathcal{F}_t = \sigma(\{W_t^H, t \geq 0\})$. One can intuitively think about fractional Wiener processes as the generalisation of the standard Wiener noise with the opportunity of configuring the structure of the noise process by adjusting the remaining amount of information from the past at time encoded into the process. It will be precisely defined in the next section. Fractional Ornstein-Uhlenbeck processes can be considered in more general form, e.g. by adding the mean-reversion parameter, or assuming that the diffusion parameter is not constant. The learning capability of the investigated neural network structures is the reason for this choice. First of all, we aim at building up accurate and consistent models for the simple cases and as the representability of the applied networks are rising, the more abstract processes will be under investigation.

Consider the drift parameter estimation of the stochastic process given by (1), which was deeply investigated by Hu and Nualart [2] in case of arbitrary Hurst exponent from the $[0,1]$ interval. The *Least Square Estimation* of the drift parameter can be written in the following form

$$\hat{\alpha}_T \doteq -\frac{\int_0^T \xi_t \, d\xi_t}{\int_0^T \xi_t^2 \, dt} = \alpha - \sigma \frac{\int_0^T \xi_t \, dW_t^H}{\int_0^T \xi_t^2 \, dt},$$

where the integral with respect to W_t^H is interpreted in Skorohod sense. If one aims at estimating the drift parameter in case of discretely observed fractional Ornstein-Uhlenbeck processes, then the error caused by the discretization of the continuous process can rise high. For discrete time series in (1) there is a consistent ergodic type estimator [2], which is defined as following:

$$\overline{\alpha}_n \doteq \left(\frac{1}{n\sigma^2 H\Gamma(H)} \sum_{k=1}^{n} \xi_{kh}^2 \right)^{-\frac{1}{2H}}, \tag{2}$$

where it is assumed that the ξ_t has been observed in discrete time points $\{t_k = kh : k = 0, 1, ..., n\}$. The following theorem shows the asymptotic consistency of the estimator given by (2).

Theorem 1. Assume the fractional Onrstein-Uhlenbeck process ξ is given by (1) is observed at discrete time points $\{t_k = kh : k = 0, 1, ..., n\}$. Suppose that h depends on the number of observations n as $n \to \infty$, h goes to 0 and nh converges to ∞. $\overline{\alpha}_n$ converges to α almost surely as $n \to \infty$, if some assumptions on h and n hold true, which can be found in [2].

The asymptotic variance of the estimator (2) is lower bounded by 4.5 and it flies away when the Hurst exponent is greater than 0.7, which reduces the applicability of the estimator, especially for small drift parameters. That is why the interest for consistent estimators rises high, with reasonable asymptotic variance, of discrete stochastic processes. We aim at investigating neural network estimators with the previous properties, hence an efficient data generator system for each transform of processes given by (1) is indispensable in this case. In case huge and complex neural network structures are teached, then a large amount of data is needed for a good performance.

3 Generator Systems for Fractional Wiener Processes

The Fractional Wiener process, which was developed by Mandelbrot [5], can be interpreted as the generalisation of the standard Wiener process in the sense that the long- and short-term dependency of the process can be adjusted by adding the Hurst exponent to the concept. The definition and some basic properties of fractional Wiener processes, needed to the investigation of simulating fractal noise, will be introduced below.

A $\{W_t^H, t \geq 0\}$ *fractional Wiener process* is defined as a centered Gaussian process with the following covariance structure

$$\operatorname{cov}(W_s^H, W_t^H) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big),$$

where the H *Hurst exponent* has to be an element of the $(0,1)$ interval.

Remark. Note that determining the mean value and the covariance structure is enough to specify the distribution of a Gaussian process, which means in this case that the distribution of a fractional Wiener process is unique for fixed Hurst exponent.

Lemma 1. As a corollary of the Kolmogorov-Chentsov continuity theorem, fractional Wiener processes have continuous modification with probability 1. Moreover, for any $\gamma \in (0,h)$ this modification is γ-Hölder continuous.

Remark. The increment process of $\left(W_t^H\right)_{t\geq 0}$ is *stationary*, i.e. $W_t^H - W_s^H \overset{d}{=} W_{t+h}^H - W_{s+h}^H$ for any $h \geq 0$. The fractional Wiener process has the H-self-similarity property, i.e. $W_{ct}^H \overset{d}{=} c^H W_t^H, t \geq 0$ for any fixed $c > 0$.

Consider the task of generating discrete trajectories of fractional Wiener processes, $\{W_t^H : t \in [0,T]\}$, for a given Hurst exponent, $H \in [0,1]$. Let me define the following operator to simplify the notations $\mathcal{T} : \mathbb{R} \times \mathbb{N} \times \mathbb{C}^1([0,\tau]) \to \mathbb{R}$, which is

$$\mathcal{T}(\tau, \mathcal{N}, \zeta(.)) \doteq \left\{\zeta(t)\mathbb{1}_{t=\frac{k\tau}{\mathcal{N}}}\right\}_{k=0}^{N}. \tag{3}$$

Therefore, the discretized noise process, $\{\hat{W}_t^H\}_{t\in[0,T]}$, with respect to the given Hurst exponent and time interval, can be obtained by taking the equidistant partition of the given time scale, $[0,T]$, and the fractional Wiener process by applying the previously defined $\mathcal{T}$ operator, i.e.

$$\{\hat{W}_t^H(\omega)\}_{t\in[0,t]} \doteq \mathcal{T}(T, N, W^H(\omega,.)).$$

Since the values $\{\hat{W}_t^H\}_{t\in[0,T]}$ form a Gaussian vector with a certain covariance matrix, they can be simulated by applying the corresponding linear transform onto a sequence of independently sampled standard normal variables.

In case of simulating fractional Wiener process, the H-self-similarity property of the matter in hand process can be taken into account to simplify the generator procedure by reducing an arbitrary time scale to the $[0,1]$ interval, i.e. a simulated $\{\hat{W}_t^H\}_{t\in[0,1]} = \left(\frac{N}{T}\right)^H \{\hat{W}_t^H\}_{t\in[0,T]}$ in distribution. This observation leads us to the amended task of generating fractional Wiener process with respect to the $[0,1]$ time scale. It is also well-known that the increments of W_t^H form a stationary Gaussian process, which leads us to the idea of rather simulating the increments and taking the cumulative sum of them to obtain the process than generating the actual process. Let me introduce an operator for taking the increments of a certain process over a partition of a given interval, $\mathcal{I} : \mathbb{R} \times \mathbb{N} \times \mathbb{R} \to \mathbb{R}^{\mathcal{N}}$, which can be formalised as

$$\mathcal{I}(\tau, \mathcal{N}, \zeta(.)) \doteq \left\{\mathcal{T}(\tau, \mathcal{N}, \zeta(.))\Big(\mathbb{1}_{t=\frac{k\tau}{\mathcal{N}}} - \mathbb{1}_{t=\frac{(k-1)\tau}{\mathcal{N}}}\Big)\right\}_{k=1}^{\mathcal{N}}. \tag{4}$$

Remark. The task of simulating discrete fractional Wiener processes with respect to the given arbitrary time scale has been reduced to generating over the unit interval, since the time attribute of the fractional Wiener process can be scaled into the unit interval by a constant, depending only on the ratio of the time interval lengths and on the corresponding Hurst exponent, thus obtaining an equality in distribution between the primary and the scaled time series. To be more precise it is enough to come up with a simulation procedure for $\mathcal{I}(1, N, W^H(\omega,.))$ because of the fact that the following equation holds

$$\left(\frac{N}{T}\right)^H \mathcal{I}(1, N, W^H(\omega,.)) = \mathcal{I}(T, N, W^H(\omega,.)).$$

Consider the covariance structure of the stationary time series given by $\{\eta_i\}_{i=1}^N \doteq I(1, N, W^H(\omega,.))$, which can be written as following by the definition of the fractional Wiener process:

$$\Psi_{W^H}^{(1,k)} \doteq \left\langle \eta_1; \eta_{k+1} \right\rangle_{\mathcal{L}^2(\Omega)} = \frac{1}{2}\Big((k+1)^{2H} + |k-1|^{2H} + 2k^{2H}\Big).$$

Since η_n is a Gaussian sequence and the covariance matrix is positive definite, (η_n) can be written as

$$\eta = \psi^{\frac{1}{2}}\varepsilon, \tag{5}$$

where $\varepsilon \doteq (\varepsilon_1, ..., \varepsilon_N)^T$ is an independently sampled vector with standard normal distribution. The equation (5) instantly leads us to the first fractional Wiener noise generator method.

3.1 Cholesky Method

Since Ψ_{W^H} is a symmetric positive definite matrix, it admits the Cholesky decomposition $\Psi_{W^H} = LL^T$, where $L = (l_{ij})_{i,j=1,...,N}$ is a lower triangular matrix, which decomposition is unique in this case, since the covariance matrix is real and positive definite. So rewrite the condition in the coordinate-wise form:

$$\sum_{k=1}^{j} l_{ik}l_{kj} = \Psi_{W^H}^{(1,i-j+1)}. \tag{6}$$

The elements of L can be computed recursively based on the previously calculated elements according to the (6) equation, i.e. l_{ij} can be determined for $i \le j$. For $i = 1$ the value of l_{11} is actually the variance, which is 1 in this case. It can be solved easily that for an arbitrary $n + 1 \ge 3$ row the elements can be determined as following:

$$l_{n+1,1} = \Psi_{W^H}^{(1,n+1)}; \quad l_{n+1,j} = l_{jj}^{-1}\left(\Psi_{W^H}^{(1,n+1-j)} - \sum_{k-1}^{j} l_{n+1,k}l_{j,k}\right).$$

As soon as the matrix L has been determined, the increment process, $(\eta_n)_{n=1}^N = \mathcal{I}(1, N, W^H(\omega, .))$, can be simulated according to the equation (5), i.e. $\eta_m = \sum_{i=1}^m \varepsilon_i l_{mi}$.

3.2 Hosking Method

Note that the square root of the Ψ_{W^H} covariance matrix can be computed more efficiently, since the matter in hand $\mathcal{I}(1, N, W^H(\omega, .))$ forms a stationary sequence, therefore the covariance structure is a Toeplitz matrix. In this subsection Hosking's method will be introduced, which was investigated for simulating stationary Gaussian sequence in a less computationally demanding way than the standard Cholesky method. Let me introduce the following notations:

$$\gamma_n \doteq \begin{pmatrix} \Psi_{W^H}^{(1,2)} \\ \Psi_{W^H}^{(1,3)} \\ \vdots \\ \Psi_{W^H}^{(1,n+1)} \end{pmatrix}, \quad J_n \doteq \mathbb{I}_n^T,$$

where $\mathbb{I}_n$ denotes the $n \times n$ dimensional identity matrix. In case $\Psi_{W^H,(N)}$ is given, the $\Psi_{W^H,(N+1)}$ covariance matrix of the sequence $\mathcal{I}(1, N+1, W^H(\omega, .))$ can be written in two forms based on the previously introduced notations:

$$\Psi_{W^H,(N+1)} = \begin{pmatrix} 1 & \gamma_n^T \\ \gamma_n & \Psi_{W^H,(N)} \end{pmatrix} = \begin{pmatrix} \Psi_{W^H,(N)} & J_n\gamma_n \\ \gamma_n^T J_n & 1 \end{pmatrix}. \tag{7}$$

Recall, that $(\eta_n)_{n=1}^{N+1}$ denoted $\mathcal{I}(1, N+1, W^H(\omega, .))$. The conditional distribution of η_{n+1} given $\eta_1, ..., \eta_n$ can be obtained by applying the normal correlation theorem, i.e.

$$\mu_n = \mathbb{E}[\eta_{n+1}|\eta_1, ..., \eta_n] = \gamma_n^T \Psi_{W^H,(N)}^{-1} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$$

$$\sigma_n^2 = \mathbb{D}^2[\eta_{n+1}|\eta_1, ..., \eta_n] = 1 - \gamma_n^T \Psi_{W^H,(N)}^{-1}\gamma_n.$$

However, calculating the inverse of $\Psi_{W^H,(N)}$ is a computationally expensive procedure, the result obtained above makes it possible to simulate $\eta_1, \eta_2, ..., \eta_n$ subsequently.

As an efficient stationary Gaussian sequence simulation, taking advantage of calculating $\Psi^{-1}_{W^H,(N)}$ recursively, Hosking's method will be described above in a slightly different form published [9]. In order to simplify the notations in the presentation of the method, let me denote $d_n \doteq \Psi^{-1}_{W^H,(N)}\gamma_n$. The second representation of $\Psi_{W^H,(N+1)}$ formalised in (7) leads us to the following equation by applying block matrix inversion:

$$\Psi^{-1}_{W^H,(N+1)} = \frac{1}{\sigma_n^2}\begin{pmatrix} \sigma_n^2\Psi^{-1}_{W^H,(N)} + J_n d_n d_n^T J_n & -J_n d_n \\ -d_n^T J_n & 1 \end{pmatrix}$$

The following recursive forms can be obtained for d_n and σ_n^2 by applying block matrix multiplication:

$$d_{n+1} = \begin{pmatrix} d_n - \phi_n J_n d_n \\ \phi_n \end{pmatrix}, \quad \sigma^2_{n+1} = \sigma_n^2 - \frac{\left(\phi_n\right)^2}{\sigma_n^2}, \tag{8}$$

where

$$t_n \doteq d_n^T J_n \gamma_n, \quad \phi_n \doteq \frac{\Psi^{(1,n+2)}_{W^H,(N)} - t_n}{\sigma_n^2},$$

where the calculation steps not presented here can be found with description in [10].

The result formalised in (8) allows us to generate $\eta_1, ..., \eta_n$ according to the method described above. For the first step, one simulates a standard normal variable to declare the first element of the sequence, η_1, then the previously introduced corresponding variables have to be calculated as follows

$$\mu_1 \doteq \Psi^{(1,2)}_{W^H,(N)}\eta_1; \quad \sigma_1^2 \doteq 1 - \left(\Psi^{(1,2)}_{W^H,(N)}\right)^2;$$
$$t_1 \doteq \left(\Psi^{(1,2)}_{W^H,(N)}\right)^2; \quad d_1 \doteq \left(\Psi^{(1,2)}_{W^H,(N)}\right).$$

Suppose that the nth values have been determined, then the $(n+1)$th element of the sequence can be obtained as $\eta_n \sim N(\mu_n, \sigma_n^2)$, while the t_n, ϕ_n and d_{n+1} auxiliary variables can be computed as described in (8). The last step of the algorithm consists of calculating the variance σ^2_{n+1} and the mean value $\mu_{n+1} \doteq d^T_{n+1}(\eta_{n+1}, ..., \eta_1)^T$ of the next element in the sequence with respect to the previously calculated auxiliary variables. As in the Cholesky method, the fractional Wiener process can be obtained as the cumulative sum of the simulated increment process $(\eta_n)_{n=1}^N = \mathcal{I}(1, N, W^H(\omega, .))$.

3.3 Davies-Harte Method

As the previous methods, the method developed by Davies and Harte [7] also focuses on finding the square root of Ψ_{W^H} by taking circulant matrix embedding into account. Note that this algorithm was proposed by Davies and Harte and was generalized by Wood and Chan [8] and later by Dietrich and Newsam [9] according to [10]. The main idea of the method is embedding the $\Psi_{W^H,(N)}$ covariance matrix in the circulant covariance matrix, $\mathcal{C}$, which algorithm is based on the following theorem [10]. Since I aim at generalising this idea to obtain a discrete isonormal integral simulator, the embedding procedure will be precisely introduced in the next section.

Theorem 2. The $\mathcal{C}$ circulant matrix with $M \times M$ dimensions has a representation $\mathcal{C} = \mathcal{Q}\Lambda\mathcal{Q}^*$, where

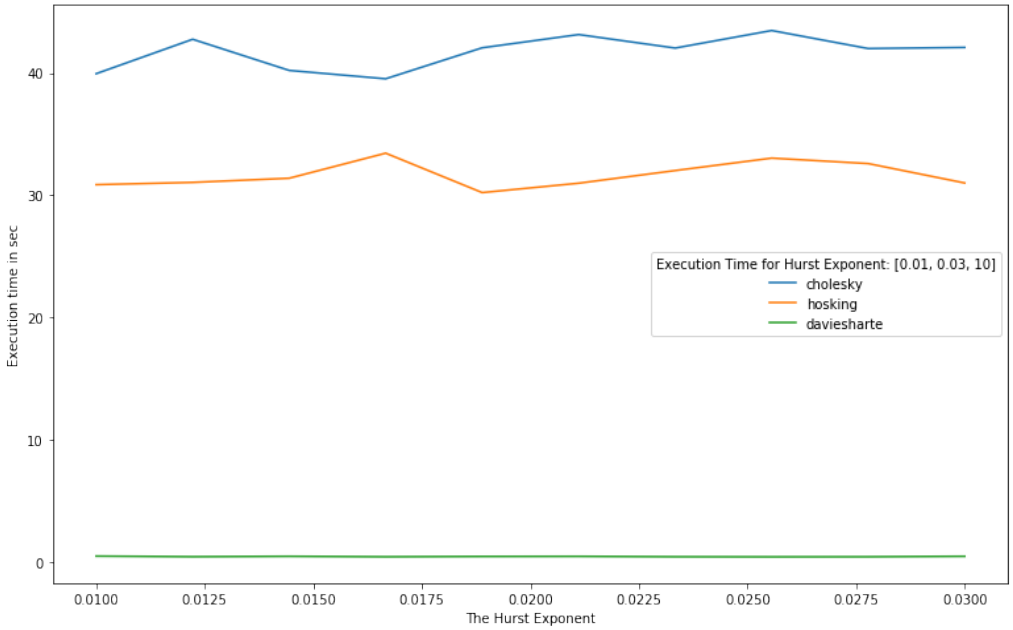
$$\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, ..., \lambda_M - 1), \quad \lambda_k = \sum_{j=0}^{M-1} c_j \exp\{-2\pi i \frac{jk}{M}\}$$

is the diagonal matrix of eigenvalues of $\mathcal{C}$, and the matrix $\mathcal{Q}$ is defined as

$$\mathcal{Q} = (q_{jk})_{j,k=0}^{M-1}, \quad q_{jk} = \frac{1}{\sqrt{M}}\exp\{-2\pi i \frac{jk}{M}\},$$

and $\mathcal{Q}^*$ denotes the conjugate transpose of $\mathcal{Q}$.

Recall, that in order to simulate Gaussian noise, one need to find the square root matrix of the covariance matrix, i.e. $\psi\psi^T = \Psi_{W^H,(N)}$, then the $\mathcal{I}(1, N, W^H(\omega, .))$ increment sequence of fractional Wiener process can be determined by multiplying the matrix ψ with a standard normal vector $(\varepsilon_1, ..., \varepsilon_N)^T$. The main advantage of this method can be derived from its complexity of order $\mathcal{O}(N\log N)$, which made it probably the most efficient among the exact methods [11]. Note that several circulant embedding methods have already been published, see e.g. [9, 12, 13].



The chart above shows the result of the execution time simulating discrete fractional Wiener processes with respect to the Hurst parameter interval $[0.01, 0.03]$, i.e. $\mathcal{I}(1, 1500, W^H(\omega, .))$, according to the introduced Cholesky, Hosking and Davies-Harte methods. As mentioned before the $[0.01, 0.03]$ interval for the Hurst exponent of fractal noise is one of the most relevant in mathematical finance, especially in the investigation of the time-dependent correlation of given stock prices. So as expected, the circulant embedding based method, in this case the Davies-Harte, is outstanding in the execution time of simulating 10 sequences with 1500 grid points over the $[0.01, 0.03]$ Hurst interval, compared to the Cholesky and Hosking methods.

4 Stochastic Integrals According to Generalised Noise

As it has been mentioned before, the main goal of my investigation in developing an efficient generator system for transformed fractional Ornstein-Uhlenbeck processes was the necessity of a fast and accurate data-generating method for teaching deep neural networks to estimate the parameters of certain Stochastic Correlation Processes. But why should be developed a limited simulating system which can generate certain fractional integrals? What would happen if the driving noise was as assumed just being isonormal process instead of fractional Wiener process. Additionally, if discretized multidimensional isonormal integrals can be simulated, then one can approximate the Wiener-Ito chaos decomposition of any square-integrable stochastic process from $\mathcal{L}^2(\Omega \times [0, T])$. These ideas will be formalised in the following subsections and also an efficient discretized isonormal integral generator method will be introduced. In order to investigate the mentioned tasks a short part of Malliavin calculus will be introduced according to [15].

4.1 Simulating Isonormal Integrals

Suppose that $\mathcal{H}$ is a separable Hilbert space with the $\langle ., . \rangle_{\mathcal{H}}$ inner product and with the $\|.\|_{\mathcal{H}}$ norm induced by the inner product.

We say that a stochastic process $\zeta = \{\zeta(h), h \in \mathcal{H}\}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is an *isonormal process* on $\mathcal{H}$ if ζ is a centered Gaussian family of random variables such that

$$\Big\langle \zeta(h), \zeta(g) \Big\rangle_{\mathcal{L}^2(\Omega)} = \big\langle h, g \big\rangle_{\mathcal{H}}. \tag{9}$$

According to Kolmogorov's theorem, for a given $\mathcal{H}$ Hilbert-space there can be constructed a complete probability space and a Gaussian process $\{\zeta(h)\}_{h\in\mathcal{H}}$ verifying the following property: the corresponding mapping $h \to \zeta(h)$ provides a linear isometry of $\mathcal{H}$ onto a closed subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ containing zero-mean Gaussian variables, which will be denoted by $\mathcal{H}_1$.

Let me consider a special case of the previously introduced mapping, since the processes - the generator system will be investigated for - belong to this special class where let $\mathcal{H}$ be a subspace of $\mathcal{L}^2([0,T])$ and let the mapping defined as

$$h \to \zeta(h) \doteq \int_0^T h(s) d\zeta_s, \tag{10}$$

where $\{\zeta(h)\}_{h\in\mathcal{H}}$ is an isonormal processes over $\mathcal{H}$. Thus, integrals for square-integrable deterministic functions with respect to isonormal process have been defined in (10).

If one aims at simulating discretized stochastic integrals with respect to an almost surely continuous process, then the issue one has to face is generating discretized pathwise Riemann-Stieltjes integrals, which simplifies the procedure to just calculating the Riemann-Stieltjes sum according to the considered stochastic integral, i.e.

$$\int_0^T h(s) d\zeta(\omega)_s \to \mathcal{I}\bigg(T, N, \int_0^T h(s) d\zeta(\omega)_s\bigg),$$

where $\mathcal{I}(\tau, \mathcal{N}, \phi(.))$ operator was defined in (4) as $\bigg\{\mathcal{T}(\tau, \mathcal{N}, \phi(.))\Big(\mathbb{1}_{t=\frac{k\tau}{\mathcal{N}}} - \mathbb{1}_{t=\frac{(k-1)\tau}{\mathcal{N}}}\Big)\bigg\}_{k=1}^{\mathcal{N}}$. Therefore, I aim at simulating $\mathcal{I}\big(T, N, \int_0^T h(s) d\zeta(\omega)_s\big)$ efficiently. Recall, that in case of simulating fractional Wiener processes, the fastest exact methods are the circulant matrix embedding based algorithms. Thus, for discretized isonormal integrals the Fast Fourier Transform and circulant embedding based methods can be generalised, so the ideas of Davies and Harte [7], Wood and Chan [8], Dietrich [9] and Kroese [13] can be applied. The method will be presented according to Dietrich's generalisation, but note that some tricks can be added, which can give a boost to the execution time. Let me denote the inner product structure of the driving noise related to the $\mathcal{I}\big(T, N, \int_0^T h(s) d\zeta(\omega)_s\big)$ time serie as following

$$\psi(k) \doteq \big\langle h_1, h_{k+1} \big\rangle_{\mathcal{H}},$$

where $\{h_k\}_{k=1}^N \doteq \mathcal{I}(T, N, \mathbb{1}_{[0,T]}(.))$. The main idea of the [7], [8], [9], [13] methods is embedding the covariance matrix in the so-called circulant matrix. In this case the inner product structure can be embedded because of the isometry between the corresponding Hilbert spaces, i.e.

$$\mathcal{C}(\psi) \doteq \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{M-2} & c_{M-1} \\ c_{M-1} & c_0 & c_1 & \dots & c_{M-3} & c_{M-2} \\ c_{M-2} & c_{M-1} & c_0 & \dots & c_{M-4} & c_{M-3} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_{M-1} & c_0 \end{pmatrix},$$

where $\mathcal{C}$ is a $2(N-1) \times 2(N-1)$ dimensional matrix, $c_0 = 1$ and $c_j \doteq \psi(k)\chi_{\{1\le k\le N-1\}} + \psi(M-k)\chi_{\{N\le k\le M\}}$. Since *Theorem* (2) the following decomposition holds true $\mathcal{C} = \mathcal{Q}\Lambda\mathcal{Q}^*$, where the matrices Λ and $\mathcal{Q}$ can be written in the following form

$$\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, ..., \lambda_M - 1), \quad \lambda_k = \sum_{j=0}^{M-1} c_j \exp\big\{-2\pi i \frac{jk}{M}\big\}, \quad q_{jk} = \frac{1}{\sqrt{M}} \exp\big\{-2\pi i \frac{jk}{M}\big\}.$$

Since this method is obtained as a generalisation of simulating any fractal noise, the practical realization of this approach requires the computation of discrete Fourier transform, both direct and inverse with regard to [10]. The multiplication by the matrix $\mathcal{Q}$ acts, up to the constant $\frac{1}{\sqrt{M}}$, as taking the discrete Fourier transform

and similarly, multiplying by the conjugate transpose of $\mathcal{Q}$, up to the constant $\sqrt{M}$, is the same operation as taking the inverse discrete Fourier transform. Without going into the details of the implementation tricks, an efficient exact method can be built up as following.

The first step is calculating the inner product structure of the driving noise over the the grid one aims at simulating, i.e. $\psi(1), ..., \psi(N-1)$ has to be calculated, where $\psi(k) = \left\langle h_1, h_{k+1} \right\rangle_{\mathcal{H}}$ and $\{h_k\}_{k=1}^{N}$ are defined as $\mathcal{I}(T, N, \mathbb{1}_{[0,T]}(.))$ according to the isometry between the two Hilbert spaces. Now one can fill the elements $c_0, ..., c_{M-1}$ needed for the embedded structure. The next step is acting the discrete Fourier transform on $(c_0, c_1, ..., c_{M-1})$ and taking the fast Fourier transform of the obtained eigenvalue vector, which has to be a real vector theoritically, but numerical imprecision can lead to negligible imaginary parts, so taking the real part of the obtained vector is suggested. In case of simulating several realizations, the steps introduced above have to calculated only once. Note, that the computation needed for generating one sequence can be reduced with some changes on the presented methods according to [13]. The third step is taking the inverse fast Fourier transform of and independent standard normal sequence $(\varepsilon_1, ..., \varepsilon_M)$ and multiplying the obtained $\frac{1}{\sqrt{M}}\mathcal{Q}^*(\varepsilon_1, ..., \varepsilon_M)$ element-wise with the square root of the vector obtained in the second step. The isonormal increments can be obtained as the real part of the fast Fourier transform of the result computed in the third step, i.e. $\mathcal{I}(T, N, \zeta(\omega, .))$ has been calculated. As the last step one has to multiply the previously determined discrete driving isonormal process element-wise with $\mathcal{T}(T, N, h(.))$ and taking the cumulative sum of the result, which leads to $\mathcal{I}\big(T, N, \int_0^T h(s) d\zeta(\omega)_s\big)$.

Note that the presented steps of the algorithm introduced above hold true without the (10) assumption, so without any additional criterions the algorithm can be implemented.

4.2 Simulating Fractional Ornstein-Uhlenbeck Processes via Isonormal Integral Generator

Fractional Ornstein-Uhlenbeck processes (1) have been introduced as the unique pathwise solution of the following stochastic differential equation with $\xi_0 \in \mathbb{R}$ initial value

$$d\xi_t = -\alpha \xi_t dt + \sigma dW_t^H,$$

where W_t^H denotes the fractional Wiener process and $\alpha, \sigma > 0$. If the initial value of the above defined differential equation is zero, then its solution can be written in the following form

$$\xi_t = -\sigma \int_0^t \mathrm{e}^{-\alpha(t-s)} dW_s^H, \tag{11}$$

which is actually a special case of the previously investigated isonormal integral according to the following observations.

Let me consider the isonormal process associated with fractional Wiener process $\{W_t^H, t \in [0,T]\}$ with respect to the corresponding Hurst exponent $H \in [0,1]$ as the centered Gaussian family $\{\zeta_H(\phi), \phi \in \mathcal{H}_H\}$, where the Hilbert space the isonormal space is defined on, $\mathcal{H}_H([0,T])$, is determined as follows. $\mathcal{H}_H$ consists of indicator functions written in the form $\phi_\tau(t) \doteq \mathbb{1}_{[0,\tau]}(t)\mathbb{1}_{[0,T]}(t)$ and the inner product, $\mathcal{H}_H([0,T])$ is endowed with, can be written as following

$$\left\langle \phi_\tau, \phi_\nu \right\rangle_{\mathcal{H}_H} \doteq \frac{1}{2}\Big(\tau^{2H} + \nu^{2H} - |\tau - \nu|^{2H}\Big), \tag{12}$$

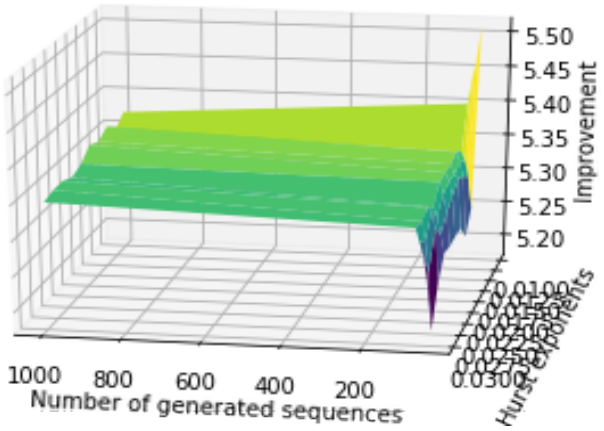
i.e. the covariance structure of the Wiener process with respect to the given H Hurst parameter. According to Kolmogorov's theorem there exists a Gaussian family over the introduced Hilbert space $\mathcal{H}_H$, $\{\zeta(\phi), \phi \in \mathcal{H}_H([0,t])\}$, which is characterized by its covariance structure

$$\left\langle \zeta(\phi_\tau), \zeta(\phi_\nu) \right\rangle_{\mathcal{L}^2(\Omega)} \doteq \left\langle \phi_\tau, \phi_\nu \right\rangle_{\mathcal{H}_H}.$$

The space of isonormal integrals can be obtained as the associated family of variables to the completion of the Hilbert space $\mathcal{H}_H([0,T])$, but in this case I omit to go into the details of this analysis.

Thus, if one aims at simulating the discretized stochastic process $\{\xi_t\}_{t\in[0,T]}$ given by (11), i.e. this task can be written as generating realisations of $\mathcal{I}\big(T, N, -\sigma \int_0^t \mathrm{e}^{-\alpha(t-s)} dW_s^H\big)$, where respect to the integral is

taken is an isonormal process, then the isonormal integral simulating method investigated in the previous section can be applied in this special case. The method has been developed with the purpose of being able to cache in memory as much data as possible. In this case the following sequence of procedures can be cached: calculation of $\mathcal{T}\big(T, N, \mathrm{e}^{-\alpha(t-s)}\big)$, determining the covariance structure of the fractional Wiener process (12), embedding the covariance structure and taking the fast Fourier transform of the discrete Fourier transformed embedded vector.



The chart above has been obtained as testing how many times faster the generalised method is able to simulate the given number of fractional Ornstein-Uhlenbeck sequences, with the given Hurst exponent from the investigated $[0.01, 0.03]$ interval over 1500 grid-points in the $[0, 1]$ time interval, then the procedure based on the fact that the driving noise simulated by Kroese's method [13]. Note that Kroese's method is also a circulant embedding based algorithm, which is four-times faster then the standard Davies-Harte method [7]. One may notice the improvement can be observed as getting smaller and smaller Hurst exponents, it is caused by the stabilisation of the inner product structure embedding on the edges of the $[0, 1]$ interval.

4.3 Implementation in Python

I have developed a Python package for simulating isonormal integrals, which consists of several fractional Wiener process generators, simulation subroutines for fractional Ornstein-Uhlenbeck processes and its transformed versions, i.e. Stochastic Correlation Processes, according to the all the driving noise simulator methods I have implemented.

However, an FBM [1] package has produced before, all the functions and class objects have been implemented in a much efficient way. I focused on investigating and developing a meta-class for simulating the elements of the first Wiener-Ito chaos with respect to isonormal noise, where the inner product structure of the noise process, the embedded inner product structure and the transformed embedded structure have been declared as self attributes. It is such an important step, since it lets me adding different caching strategies for the mentioned calculations, which could give a huge boost for the execution time. A *getitem*() subroutine has been written for simulating the actual increments of the given fractal noise by applying the previously calculated transformed embedded inner product structure according to multiplying with certain Fourier transformed standard normal variables. So if one aims at simulating many sequences with respect to the given parameters then the introduced class object has to be declared and only the *getitem*() subroutine has to be called to get the given number of sequences. For example, the class for generating fractional Ornstein-Uhlenbeck processes can be deduced from the mentioned meta-class as declaring the weights, the increments will be multiplied element-wise, according to the deterministic integrand related to the fractional Ornstein-Uhlenbeck processes. Note that the meta-class has the property that it can be used as data loader object of Pytorch, which makes it easy to apply the meta-class as a data-generating procedure for neural networks developed in Pytorch.

[1] `https://pypi.org/project/fbm/`

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