Simulating Transformed Fractional Ornstein-Uhlenbeck Processes

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Motivation

Stochastic Correlation Processes

$$d\xi_t = \kappa\big(\mu - \tanh(\xi_t)\big)dt + \sigma dW_t^H$$
$$d\rho_t = \tanh(\xi_t)$$

- Parameter estimation of Stochastic Correlation Processes (SCP)
- Too large asymptotic variance even for fractional Ornstein-Uhlenbeck processes
- Applying deep learning based estimators instead of classical statistic methods
- Necessity of an efficient data-generating system in Python with compatibility of Pytorch's dataloader metaclass

Fractional Wiener Processes I.

Fractional Wiener Process

A $\{W_t^H, t \geq 0\}$ *fractional Wiener process* is defined as a centered Gaussian process with the following covariance structure

$$\langle W_s^H, W_t^H \rangle_{\mathcal{L}^2(\Omega)} = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

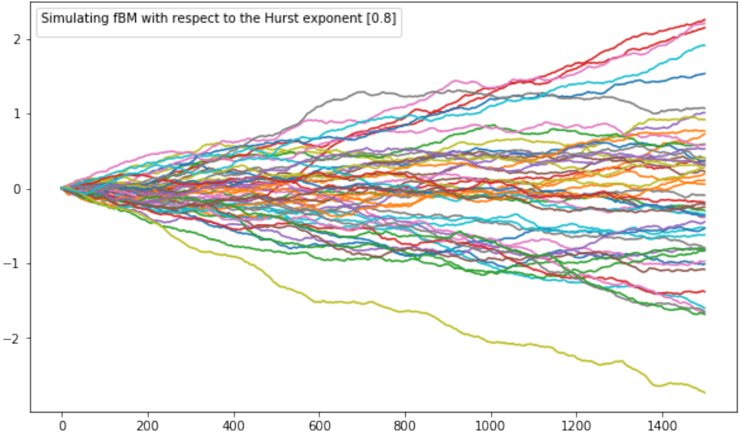
where the H *Hurst exponent* has to be an element of the $[0,1]$ interval.

The learning task

The set of parameters has been reduced to the Hurst exponent alone, additionally fractional Wiener process will be applied as driving noise in the upcoming stochastic processes.

Fractional Wiener Processes II.

Fractional Wiener Processes III.



Simulating Fractal Noise

Gaussian process

If $(\eta_n)_{n\in\mathcal{N}}$ is a Gaussian sequence with the covariance matrix $\Psi = \Gamma\Gamma^T$ then $\{\eta_j\}_{j=1}^M$ can be simulated as following where $\varepsilon = \{\varepsilon_j\}_{j=1}^M$ is an independent standard normal sample

$$\eta = \Gamma\varepsilon.$$

Fractional Wiener Generators

All methods focus on finding the square root of the covariance matrix efficiently

- Cholesky method (according to the Cholesky decomposition)
- Hosking method (because of the stationary increments, the covariance structure is a Toeplitz matrix)
- Davies-Harte method (circulant embedding based algorithm)

Fractional Ornstein-Uhlenbeck Processes I.

Let me consider the fractional Ornstein-Uhlenbeck process as the pathwise unique solution of the following stochastic differential equation with $\xi_0 \in \mathbb{R}$ initial value

$$d\xi_t = -\alpha\xi_t dt + \sigma dW_t^H,$$

where the σ diffusion parameter and the α drift parameter have to be positive real numbers, W_t^H with $H \in [0,1]$ Hurst exponent is a fractional Wiener process in its natural filtration.

If one aims at simulating a discretized fOU process then the following parts have to be discretized and generated

- a Lebesgue-integral $(-\alpha\xi_t dt)$
- a pathwise Riemann-Stieltjes integral (σdW_t^H)

Fractional Ornstein-Uhlenbeck Processes II.

If the initial value of the previously introduced differential equation is zero, then its solution can be written in the following form

$$\xi_t = -\sigma \int_0^t \mathrm{e}^{-\alpha(t-s)} dW_s^H,$$

which is actually an element of the first Wiener-Ito chaos with respect to the fractional Wiener process.

- Simulating efficiently the obtained stochastic integral, i.e. an element of the first Wiener-Ito chaos
- Can it be generalised for the p-times tensor product of the deterministic integrands from $\mathcal{L}^2([0,T])$?
- Which are the necessary conditions to decompose $\mathcal{L}^2(\Omega \times [0,T])$ according to the W-I chaos decomposition?

Isonormal Processes I.

We say that a stochastic process $\zeta = \{\zeta(h), h \in \mathcal{H}\}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is an isonormal process on $\mathcal{H}$ if ζ is a centered Gaussian family of random variables such that

$$\left\langle \zeta(h), \zeta(g) \right\rangle_{\mathcal{L}^2(\Omega)} = \left\langle h, g \right\rangle_{\mathcal{H}}.$$

- $\phi_\tau(t) \doteq \mathbb{1}_{[0,\tau]}(t)\mathbb{1}_{[0,T]}(t) \in \mathcal{H}_H([0,T])$ and $\mathcal{H}_H$ is endowed with the inner product

$$\left\langle \phi_\tau, \phi_\nu \right\rangle_{\mathcal{H}_H} \doteq \frac{1}{2}\left(\tau^{2H} + \nu^{2H} - |\tau - \nu|^{2H}\right).$$

Isonormal Processes II.

- According to Kolmogorov's theorem there exists a Gaussian family over the introduced Hilbert space $\mathcal{H}_H$, $\{\zeta(\phi), \phi \in \mathcal{H}_H([0,T])\}$, which is characterized by its covariance structure

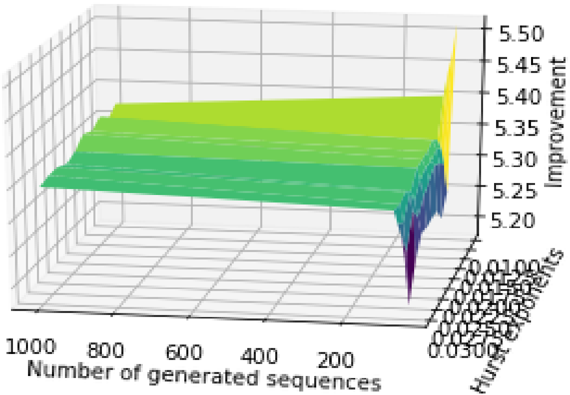
$$\left\langle \zeta(\phi_\tau), \zeta(\phi_\nu) \right\rangle_{\mathcal{L}^2(\Omega)} \doteq \left\langle \phi_\tau, \phi_\nu \right\rangle_{\mathcal{H}_H}.$$

- Let $\mathcal{C}_H([0,T])$ be the linear closure of $\mathcal{H}_H([0,T])$
- The discretized fOU serie is an element of $\mathcal{C}_H([0,T])$ with the corresponding Hurst exponent

Implementation

- An FBM package has been published before on Github (Cholesky, Hosking, Davies-Harte), where all of them have been re-implemented
- fOU and SCP generators for arbitrary driving noise, parameter set and initial value
- A metaclass for simulating isonormal processes by generalising Kroese's idea, special case: fOU
- Caching strategy can be applied on the 60% of the computations needed for one realisation

Simulating fOU With Respect to Isonormal Process



Further Work

- Formalising the idea for multidimensional isonormal processes obtained as the image of the tensor product of functions from $\mathcal{C}_H([0,T])$
- Implementing the multidimensional case
- Each subspace of the W-I chaos decomposition can be simulated, what can be the application of such a generator system
- May be implementing Malliavin calculus for my system, only if it has applications, e.g. in finance

Thank you for your attention!