Piacok árazása matroidokkal adott kiértékelési függvények esetén

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Az előző félévekben elkezdett dinamikus árazási témával foglalkoztam ebben a félévben is. Az első két félévben a dinamikus árazás egy új megközelítésével sikerült a bi-demand esetben egy új eredményt elérni, miszerint ha minden vásárló igénye legfeljebb 2, akkor van polinomiális időben kiszámítható dinamikus árazás, mely eléri az optimális közjólét értékét. Azonban az első két félévben használtuk azt a feltételt, hogy minden optimális megoldásban minden vásárló annyi tárgyat kap, amennyi az igénye. A harmadik félévben ezt az eredményt erősítettük azzal, hogy elhagytuk ezt a feltételt. A félév fő eredmény pedig annak az igazolása, hogy tetszőleges számú vásárló esetén akkor is van polinomiális időben kiszámítható optimális dinamikus árazás, ha minden vásárló kiértékelési függvénye 2-rangú matroid rangfüggvénye. Ez utóbbi bizonyítása nagyon hasonlít a bi-demand eset bizonyításához, így a beszámoló csak a bi-demand eset részletes tárgyalását tartalmazza.

A dual approach for dynamic pricing in multi-demand markets*

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1 Introduction

A combinatorial market consists of a set of indivisible goods and a set of buyers, where each buyer has a valuation function that represents the buyer's preferences over the subsets of items. From an optimization point of view, the goal is to find an allocation of the items to buyers in such a way that the total sum of the buyers' values is maximized – this sum is called the social welfare. An optimal allocation can be found efficiently in various settings [8, 17, 23, 25], but the problem becomes significantly more difficult if one would like to realize the optimal social welfare in an automatic way through simple mechanisms.

A great amount of work concentrated on finding optimal pricing schemes. Given a price for each item, we define the utility of a buyer for a bundle of items to be the value of the bundle with respect to the buyer's valuation, minus the total price of the items in the bundle. A pair of pricing and allocation is called a Walrasian equilibrium if the market clears (that is, all the items are assigned to buyers) and everyone receives a bundle that maximizes her utility. Given any Walrasian equilibrium, the corresponding price vector is referred to as Walrasian pricing, and the definition implies that the corresponding allocation maximizes social welfare.

Although Walrasian equilibria have distinguished properties, Cohen-Addad et al. [9] realized that the existence of a Walrasian equilibrium alone is not sufficient to achieve optimal social welfare based on buyers' decisions. Different bundles of items might have the same utility for the same buyer, and in such cases ties must be broken by a central coordinator in order to ensure that the optimal social welfare is achieved. However, the presence of such a tie-breaking rule is unrealistic in real life markets and buyers choose an arbitrary best bundle for themselves without caring about social optimum.

Dynamic pricing schemes were introduced as an alternative to posted-price mechanisms that are capable of maximizing social welfare even without a central tie-breaking coordinator. In this model, the buyers arrive in a sequential order, and each buyer selects a bundle of the remaining items that maximizes her utility. The buyers' preferences are known in advance, and the seller is allowed to update the prices between buyer arrivals based upon the remaining set of items, but without knowing the identity of the next buyer. The main open problem in [9] asked whether any market with gross substitutes valuations has a dynamic pricing scheme that achieves optimal social welfare.

Related work Walrasian equilibria were introduced already in the late 1800s [26] for divisible goods. A century later, Kelso and Crawford [20] defined gross substitutes functions and verified

*A preliminary version of the work appeared on ArXiv [1].

the existence of Walrasian prices for such valuations. It is worth mentioning that the class of gross substitutes functions coincides with that of M$^\natural$-concave functions, introduced by Murota and Shioura [22]. The fundamental role of the gross substitutes condition was recognized by Gul and Stacchetti [18] who verified that it is necessary to ensure the existence of a Walrasian equilibrium.

Cohen-Addad et al. [9] and independently Hsu et al. [19] observed that Walrasian prices are not powerful enough to control the market on their own. The reason is that ties among different bundles must be broken in a coordinated fashion that is consistent with maximizing social welfare. Furthermore, this problem cannot be resolved by finding Walrasian prices where ties do not occur as [19] showed that minimal Walrasian prices necessarily induce ties. To overcome these difficulties, [9] introduced the notion of dynamic pricing schemes, where prices can be redefined between buyer-arrivals. They proposed a scheme maximizing social welfare for matching or unit-demand markets, where the valuation of each buyer is determined by the most valuable item in her bundle. In each phase, the algorithm constructs a so-called 'relation graph' and performs various computations upon it. Then the prices are updated based on structural properties of the graph.

Recently, Berger et al. [3] considered markets beyond unit-demand valuations, and provided a clever polynomial-time algorithm for finding optimal dynamic prices up to three multi-demand buyers. Their approach is based on a generalization of the relation graph of [9] that they call a 'preference graph', and on a new directed graph termed the 'item-equivalence graph'. They showed that there is a strong connection between these two graphs, and provided a clever pricing scheme based on these observations.

Further results on posted-price mechanisms considered matroid rank valuations [2], relaxations such as combinatorial Walrasian equilibrium [16], and online settings [4–7, 10–12, 14, 15].

Our contribution In the present work, we focus on multi-demand combinatorial markets. In this setting, each buyer t has a positive integer bound $b(t)$ on the number of desired items, and the value of a set is the sum of the values of the $b(t)$ most valued items in the set. In particular, if we set each $b(t)$ to one then we get back the unit-demand case.

For multi-demand markets, the problem of finding an allocation that maximizes social welfare is equivalent to a maximum weight b-matching problem in a bipartite graph with vertex classes corresponding to the buyers and items, respectively. Note that, unlike in the case of Walrasian equilibrium, clearing the market is not required as a maximum weight b-matching might leave some of the items unallocated. The high level idea of our approach is to consider the dual of this problem, and to define an appropriate price vector based on an optimal dual solution with distinguished structural properties.

Based on the primal-dual interpretation of the problem, we give a simpler proof of a result of Cohen-Addad et al. [9] on unit-demand valuations. Although this can be considered a special case of bi-demand markets, we discuss it separately as an illustration of our techniques.

Theorem 1 (Cohen-Addad et al.)**.** *Every unit-demand market admits an optimal dynamic pricing that can be computed in polynomial time.*

The problem becomes significantly more difficult for larger demands. Berger et al. [3] observed that bundles that are given to a buyer in different optimal allocations satisfy strong structural properties. For markets up to three multi-demand buyers, they grouped the items into at most eight equivalence classes based on which buyer could get them in an optimal solution, and then analyzed the item-equivalence graph for obtaining an optimal dynamic pricing. We show that these properties follow from the primal-dual interpretation of the problem, and give a new proof of their result.

Theorem 2 (Berger et al.)**.** *Every multi-demand market up to three buyers admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

The main result of this work is an algorithm for determining optimal dynamic prices for bi-demand markets with an arbitrary number of buyers, that is, when the demand $b(t)$ is two for each buyer t. Besides structural observations on the dual solution, the proof relies on uncrossing sets that are problematic in terms of resolving ties.

Theorem 3. *Every bi-demand market admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

When the total demand of buyers exceeds the number of available items, ensuring the optimality of the final allocation becomes more intricate. Therefore, first we consider instances satisfying the following property:

(OPT) each buyer $t \in T$ receives exactly $b(t)$ items in every optimal allocation.

While this is a restrictive assumption, it is a reasonable condition that holds for a wide range of applications. For example, if the total number of items is not less than the total demand of the buyers and the value of each item is strictly positive for each buyer, then it is not difficult to check that (OPT) is satisfied. We prove Theorems 2 and 3 under assumption (OPT) first; the proofs of the general cases are rather technical and so are deferred to the Appendix.

This work is organized as follows. Basic definitions and notation are given in Section 2, while Section 3 provides structural observations on optimal dynamic prices in multi-demand markets. Unit demand markets and multi-demand markets up to three buyers satisfying the the (OPT) condition are discussed in Section 4. Finally, Section 5 solves the bi-demand case, also under the (OPT) condition. In the Appendix, we prove the existence dynamic prices in multi-demand markets up to three buyers and in the bi-demand case without assuming (OPT).

2 Preliminaries

Basic notation. We denote the sets of *real*, *non-negative real*, *integer*, and *positive integer* numbers by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, and $\mathbb{Z}_{>0}$, respectively. Given a ground set S and subsets $X, Y \subseteq S$, the *difference* of X and Y is denoted by $X - Y$. If Y consists of a single element y, then $X - \{y\}$ and $X \cup \{y\}$ are abbreviated by $X - y$ and $X + y$, respectively. The *symmetric difference* of X and Y is $X \triangle Y := (X - Y) \cup (Y - X)$. For a function $f : S \to \mathbb{R}$, the total sum of its values over a set X is denoted by $f(X) := \sum_{s \in X} f(s)$. The *inner product* of two vectors $x, y \in \mathbb{R}^S$ is $x \cdot y := \sum_{s \in S} x(s)y(s)$. Given a set S, an *ordering* of S is a bijection between S and the set of integers $\{1, \dots, |S|\}$. For a set $X \subseteq S$, we denote the restriction of the ordering to $S - X$ by $\sigma|_{S-X}$. Given orderings σ_1 and σ_2 of disjoint sets S_1 and S_2, respectively, we denote by $\sigma = (\sigma_1, \sigma_2)$ the ordering of $S := S_1 \cup S_2$ where $\sigma(s) = \sigma_1(s)$ for $s \in S_1$ and $\sigma_2(s) + |S_1|$ for $s \in S_2$.

Let $G = (S, T; E)$ be a bipartite graph with vertex classes S and T and edge set E. We will always denote the *vertex set* of the graph by $V := S \cup T$. For a subset $X \subseteq V$, we denote the *set of edges induced by* X by $E[X]$, while $G[X]$ stands for the *graph induced by* X. The graph obtained from G by *deleting* X is denoted by $G - X$. Given a subset $F \subseteq E$, the *set of edges in* F *incident to a vertex* $v \in V$ is denoted by $\delta_F(v)$. Accordingly, the *degree* of v in F is $d_F(v) := |\delta_F(v)|$. For a set $Z \subseteq T$, the *set of neighbors of* Z *with respect to* F is denoted by $N_F(Z)$, that is, $N_F(Z) := \{s \in S \mid \text{there exists and edge } st \in F \text{ with } t \in Z\}$. The subscript F is dropped from the notation or is changed to G whenever F is the whole edge set.

Market model. A combinatorial market consists of a set S of *indivisible items* and a set T of *buyers*. We consider *multi-demand*[1] markets, where each buyer $t \in T$ has a valuation $v_t : S \to \mathbb{R}_+$ over individual items together with an upper bound $b(t)$ on the number of desired items, and the *value* of a set $X \subseteq S$ for buyer t is defined as $v_t(X) := \max\{v_t(X') \mid X' \subseteq X, |X'| \leq b(t)\}$. *Unit-demand* and *bi-demand* valuations correspond to the special cases when $b(t) = 1$ and $b(t) = 2$ for each $t \in T$, respectively.

Given a *price vector* $p : S \to \mathbb{R}_+$, the *utility* of buyer t for X is defined as $u_t(X) := v_t(X) - p(X)$. The buyers, whose valuations are known in advance, arrive in an undetermined order, and the next buyer always chooses a subset of at most her desired number of items that maximizes her utility. In contrast to static models, the prices can be updated between buyer-arrivals based on the remaining sets of items and buyers. The goal is to set the prices at each phase in such a way that no matter in what order the buyers arrive, the final allocation maximizes the social welfare. Such a pricing scheme and allocation are called *optimal*. It is worth emphasizing that a buyer may decide either to take or not to take an item which has 0 utility, that is, it might happen that the bundle of items that she chooses is not inclusionwise minimal. This seemingly tiny degree of freedom actually results in difficulties that one has to take care of.

We may assume that all items are allocated in every optimal allocation, therefore $|S| \leq \sum_{t \in T} b(t)$. Indeed, if we take in optimal allocation that uses a minimum number of items, then we can set the price of unused items to a large value so that no buyer takes them. In particular, when (OPT) is assumed, then the number of items coincides with the total demand of the buyers.

3 Optimal allocations and maximum weight b-matchings

A combinatorial market with multi-demand valuations can be naturally identified with an edge-weighted complete bipartite graph $G = (S, T; E)$ where S is the set of items, T is the set of buyers, and for every item s and buyer t the weight of edge $st \in E$ is $w(st) := v_t(s)$. We extend the demands to S as well by setting $b(s) = 1$ for every $s \in S$. Then an optimal allocation of the items corresponds to a maximum weight subset $M \subseteq E$ satisfying $d_M(v) \leq b(v)$ for each $v \in S \cup T$.

Let $G = (S, T; E)$ be a bipartite graph and recall that $V := S \cup T$. Given an upper bound $b : V \to \mathbb{Z}_+$ on the vertices, a subset $M \subseteq E$ is called a *b-matching* if $d_M(v) \leq b(v)$ for every $v \in V$. If equality holds for each $v \in V$, then M is called a *b-factor*. Notice that if $b(v) = 1$ for each $v \in V$, then a b-matching or b-factor is simply a matching or perfect matching, respectively. Kőnig's classical theorem [21] gives a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph.

Theorem 4 (Kőnig)**.** *There exists a perfect matching in a bipartite graph $G = (S, T; E)$ if and only if $|S| = |T|$ and $|N(Y)| \geq |Y|$ for every $Y \subseteq T$.*

Let $w : E \to \mathbb{R}$ be a weight function on the edges. A function $\pi : V \to \mathbb{R}$ on the vertex set $V = S \cup T$ is a *weighted covering* of w if $\pi(s) + \pi(t) \geq w(st)$ holds for every edge $st \in E$. An edge st is called *tight with respect to* π if $\pi(s) + \pi(t) = w(st)$. The *total value* of the covering is $\pi \cdot b = \sum_{v \in V} \pi(v) \cdot b(v)$. We refer to a covering of minimum total value as *minimum weighted covering*. The celebrated result of Egerváry [13] provides a min-max characterization for the maximum weight of a perfect matching in a bipartite graph.

[1] Multi-demand valuations are special cases of weighted matroid rank functions for uniform matroids, see [2].

Theorem 5 (Egerváry)**.** *Let $G = (S, T; E)$ be a graph, $w : W \to \mathbb{R}$ be a weight function. Then the maximum weight of a matching is equal to the minimum total value of a non-negative weighted covering π of w. If G has a perfect matching, then the maximum weight of a perfect matching is equal to the minimum total value of a weighted covering π of w.*

In general, a b-factor or even a maximum weight b-matching can be found in polynomial time (even in non-bipartite graphs, see e.g. [24]). However, when b is identically one on S, then a characterization follows easily from Kőnig's and Egerváry's theorems[2].

Theorem 6. *Let $G = (S, T; E)$ be a bipartite graph, $w : E \to \mathbb{R}_+$ be a weight function, and $b : V \to \mathbb{Z}_{>0}$ be an upper bound function satisfying $b(s) = 1$ for $s \in S$.*

(a) G has a b-factor if and only if $|S| = b(T)$ and $|N(X)| \geq b(X)$ for every $X \subseteq T$.

(b) The maximum w-weight of a b-matching is equal to the minimum total value of a non-negative weighted covering π of w.

Proof. Let $G' = (S', T; E')$ denote the graph obtained from G by taking $b(t)$ copies of each vertex $t \in T$ and connecting them to the vertices in $N_G(t)$. It is not difficult to check that G has a b-factor if and only if G' has a perfect matching, thus first part of the theorem follows by Theorem 4.

To see the second part, for each copy $t' \in T'$ of an original vertex $t \in T$, define the weight of edge st' as $w'(st') := w(st)$. Then the maximum w-weight of a b-matching of G is equal to the maximum w'-weight of a matching of G'. Now take an optimal weighted covering π' of w' in G'. As the different copies of an original vertex $t \in T$ share the same neighbors in G', each of them receive the same value in any optimal weighted covering of w' - define $\pi(t)$ to be this value. Then π is a weighted covering of w in G with total value equal to that of π', hence the theorem follows by Theorem 5. $\square$

Given a weighted cover π, the *subgraph of tight edges* with respect to π is denoted by $G_\pi = (S, T; E_\pi)$. In what follows, we prove some easy structural results on the relation of optimal b-matchings and weighted coverings.

Lemma 1. *Let $G = (S, T; E)$ be a bipartite graph, $w : E \to \mathbb{R}_+$ be a weight function, and $b : V \to \mathbb{Z}_{>0}$ be an upper bound function satisfying $b(s) = 1$ for $s \in S$. Then $M \subseteq E_\pi$ holds for any pair of maximum weight b-matching M and minimum weighted covering π, and for a vertex $v \in V$ we have $\pi(v) = 0$ if $d_M(v) < b(v)$. For b-factors, the reverse implication also holds, that is, M is a maximum w-weight b-factor in G if and only if M is a b-factor in G_π for some minimum weighted covering π.*

Proof. Let M be a maximum weight b-matching and π be a non-negative minimum weighted cover. We have $w(M) = \sum_{st \in M} w(st) \leq \sum_{st \in M}(\pi(s) + \pi(t)) \leq \sum_{v \in V} \pi(v) \cdot b(v)$, and if equality holds throughout, then M necessarily consists of tight edges and $\pi(v) = 0$ if $d_M(v) < b(v)$.

Now consider the b-factor case. Let M be a maximum weight b-factor and π be a minimum weighted cover. We have $w(M) = \sum_{st \in M} w(st) \leq \sum_{st \in M}(\pi(s) + \pi(t)) = \sum_{v \in V} \pi(v) \cdot b(v)$, and the inequality is satisfied with equality if and only if M consists of tight edges. $\square$

Following the notation of [3], we call an edge $st \in E$ *legal* if there exists a maximum weight b-matching containing it, and say that s is *legal for* t. A subset $F \subseteq \delta(t)$ is *feasible* if there exists a maximum weight b-matching M such that $\delta_M(t) = F$; in this case $N_F(t)$ is called *feasible for* t[3]. Notice that a feasible set necessarily consists of legal edges. The essence of the following

[2]The same results follow by strong duality applied to the linear programming formulations of the problems.

[3]The notion of feasibility is closely related to 'legal allocations' introduced in [3]. However, 'legal subsets' are different from feasible ones, hence we use a different term here to avoid confusion.

technical lemma is that there exists a minimum weighted covering for which G_π consists only of legal edges, thus giving a better structural understanding of optimal dual solutions; for an illustration see Figure 1.

Lemma 2. *The optimal π attaining the minimum in Theorem 6(b) can be chosen such that*

(a) an edge st is tight with respect to π if and only if it is legal, and

(b) $\pi(v) = 0$ for some $v \in V$ if and only if there exists a maximum weight b-matching M with $d_M(v) < b(v)$.

Furthermore, such a π can be determined in polynomial time.

Proof. In both cases, the 'if' part follows by Lemma 1. Let M and π be a maximum weight b-matching and a minimum weighted covering, respectively. To prove the lemma, we will modify π in two phases.

In the first phase, we ensure (a) to hold. Take an arbitrary ordering $e_1, \ldots, e_m$ of the edges, and set $\pi_0 := \pi$ and $w_0 := w$. For $i = 1, \ldots, m$, repeat the following steps. Let $\varepsilon_i := \max\{w_{i-1}(M) \mid M \text{ is a } b\text{-matching}\} - \max\{w_{i-1}(M) \mid M \text{ is a } b\text{-matching containing } e_i\}$. Let w_i denote the weight function obtained from w_{i-1} by increasing the weight of e_i by $\varepsilon_i/2$, and let π_i be a minimum weighted covering of w_i. Due to the definition of ε_i, a b-matching M has maximum weight with respect to w_i if and only if it has maximum weight with respect to w_{i-1}, and in this case $w_i(M) = w_{i-1}(M)$. That is, the sets of maximum weight b-matchings with respect to w and w_m coincide, and the weight of legal edges does not change, therefore π_m is a minimum weighted covering of w as well.

In the second phase, we concentrate on (b). Take an arbitrary ordering $v_1, \ldots, v_n$ of the vertices, and consider π_m and w_m that the previous phase stopped with. For $j = 1, \ldots, n$, repeat the following steps. Let $\delta_j := \max\{w_{m+j-1}(M) \mid M \text{ is a } b\text{-matching}\} - \max\{w_{m+j-1}(M) \mid M \text{ is a } b\text{-matching}, d_M(v_j) \leq b(v_j) - 1\}$. Let w_{m+j} denote the weight function obtained from w_{m+j-1} by decreasing the weight of the edges incident to v_j by $\delta_j/(b(v_j)+1)$, and let π_{m+j} be a minimum weighted covering of w_{m+j}. Due to the definition of δ_j, a b-matching M has maximum weight with respect to w_{m+j-1} if and only if it has maximum weight with respect to w_{m+j}, and in this case $w_{m+j}(M) = w_{m+j-1}(M) - \delta_j \cdot b(v_j)$. That is, the sets of maximum weight b-matchings with respect to w and w_{m+n} coincide. Let π' denote the weighted covering of w obtained by increasing the value of $\pi_{m+n}(v_\ell)$ by $\delta_\ell/(b(v_\ell) + 1)$ for $\ell = 1, \ldots, n$. As the total value of π' is greater than that of π_{m+n} by exactly $\max\{w(M) \mid M \text{ is a } b\text{-matching}\} - \max\{w_{m+n}(M) \mid M \text{ is a } b\text{-matching}\}$, π' is a minimum weighted covering of w.

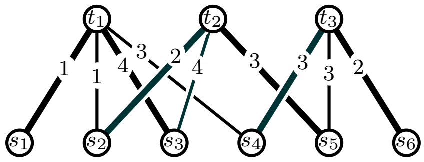
As $\varepsilon_i > 0$ whenever e_i is not legal and $\delta_j > 0$ whenever there exists a maximum weight b-matching M with $d_M(v_j) < b(v_j)$, π' satisfies both (a) and (b) as required.

□

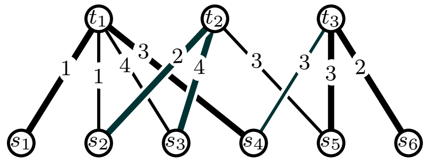
Remark 7. If the market satisfies property (OPT), the lemma implies that there exists a minimum weight cover that is positive for every buyer and every item.

Feasible sets play a key role in the design of optimal dynamic pricing schemes. After the current buyer leaves, the associated bipartite graph is updated by deleting the vertices corresponding to the buyer and her bundle of items, and the prices are recomputed for the remaining items. It follows from the definitions that the scheme is optimal if and only if the prices are always set in such a way that any bundle of items maximizing the utility of an agent t forms a feasible set for t.

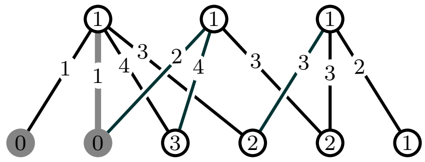
The high-level idea of our approach is as follows. First, we take a minimum weighted cover π provided by Lemma 2. If we define the price of an item $s \in S$ to be $\pi(s)$, then for any $t \in T$ we have $u_t(s) = v_t(s) - \pi(s) = w(st) - \pi(s) \leq \pi(t)$ and, by Lemma 2(a), equality holds if



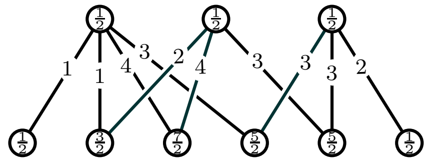
(a) Maximum weight b-matching $M_1 = \{t_1s_1, t_1s_3, t_2s_2, t_2s_5, t_3s_4, t_3s_6\}$.



(b) Maximum weight b-matching $M_2 = \{t_1s_1, t_1s_4, t_2s_2, t_2s_3, t_3s_5, t_3s_6\}$



(c) A minimum weighted covering π. Notice that s_1t_1 is tight but not legal, and $\pi(s_1) = \pi(s_2) = 0$ although $d_M(s_1) = d_M(s_2) = 1$ for every maximum weight b-matching.



(d) A minimum weighted covering satisfying the conditions of Lemma 2.

Figure 1: A bipartite graph corresponding to a market with three buyers having demand two and six items. The numbers denote the weights of the edges; all the remaining edges have weight 0. There are two maximum weight b-matchings M_1 (Figure 1a) and M_2 (Figure 1b). Notice that both s_3t_1 and s_4t_1 are legal, but they do not form a feasible set.

and only if s is feasible for t. This means that each buyer prefers choosing items that are legal for her. For unit-demand valuations, such a solution immediately yields an optimal dynamic pricing scheme. However, when the demands are greater than one, a collection of legal items might not form a feasible set, see an example on Figure 1. In order to control the choices of the buyers, we slightly perturb the item prices by choosing an ordering $\sigma : S \to \{1, \dots, |S|\}$ and set the price of item s to be $\pi(s) + \delta \cdot \sigma(s)$ for some sufficiently small $\delta > 0$. Here the value of $\sigma(s)$ will be set in such a way that any bundle of items maximizing the utility of a buyer will form a feasible set for her, as needed.

Given a bipartite graph $G = (S, T; E)$ and upper bounds $b : V \to \mathbb{Z}_{>0}$ with $b(s) = 1$ for $s \in S$, we call an ordering $\sigma : S \to \{1, \dots, |S|\}$ *adequate for G* if it satisfies the following condition: for any $t \in T$, there exists a b-factor in G that matches t to its first $b(t)$ neighbors according to the ordering σ. For ease of notation, we introduce the *slack* of π to denote

$$\Delta(\pi) := \min\big\{\min\{\pi(t) + \pi(s) - w(st) \mid st \in E, st \text{ is not tight}\},\ \min\{\pi(v) \mid v \in V, \pi(v) > 0\}\big\},$$

where the minimum over an empty set is defined to be $+\infty$. Using this terminology, the above idea is formalized in the following lemma.

Lemma 3. *Assume that* (OPT) *is satisfied. Let $G = (S, T; E)$ be the edge-weighted bipartite graph associated with the market, π be a weighted cover provided by Lemma 2, and σ be an adequate ordering for G_π. For $\delta := \Delta(\pi)/(|S| + 1)$, setting the prices to $p(s) := \pi(s) + \delta \cdot \sigma(s)$ results in optimal dynamic prices.*

Proof. By (OPT), every optimal solution is a b-factor. Observe that for any $s \in S$ and $t \in T$, we have

$$\begin{aligned} u_t(s) &= v_t(s) - p(s) \\ &= w(st) - (\pi(s) + \delta \cdot \sigma(s)) \\ &\le \pi(t) - \delta \cdot \sigma(s). \end{aligned}$$

Here equality holds if and only if st is tight with respect to π, in which case $u_t(s) = \pi(t) - \delta \cdot \sigma(s) > \pi(t) - \Delta(\pi) \cdot |S|/(|S|+1) > 0$ by the choice of δ and by Lemma 2(b). Furthermore, if st is tight and $s't$ is a non-tight edge of G, then $u_t(s') \leq \pi(t) - \Delta(\pi) \leq \pi(t) - \delta(|S|+1) < u_t(s)$ by the choice of δ. Concluding the above, we get that no matter which buyer arrives next, she strictly prefers legal items over non-legal ones, and legal items have strictly positive utility values for her. That is, she chooses the first $b(t)$ of its neighbors in G_π according to the ordering σ. As σ is adequate for G_π, the statement follows by Lemma 1. ☐

4 Unit- and multi-demand markets

4.1 Unit-demand markets

The existence of optimal dynamic prices for unit-demand valuations was settled in [9]. As an illustration of our approach, we give a simple algorithm that uses an optimal dual solution.

Theorem 1 (Cohen-Addad et al.). *Every unit-demand market admits an optimal dynamic pricing that can be computed in polynomial time.*

Proof. Consider the bipartite graph associated with the market, take an optimal cover π provided by Lemma 2, and set the price of item s to be $\pi(s)$. For a pair of buyer $t \in T$ and $s \in S$, we have

$$\begin{aligned} u_t(s) &= v_t(s) - p(s) \\ &= w(st) - p(s) \\ &\leq (\pi(s) + \pi(t)) - \pi(s) \\ &= \pi(t). \end{aligned}$$

By Lemma 2(a), strict equality holds if and only if st is legal. We claim that no matter which buyer arrives next, she either chooses an item that is legal (and so forms a feasible set for her), or she takes none of the items and the empty set is feasible for her.

To see this, assume first that $\pi(t) > 0$. By Lemma 2(b), there exists at least one item legal for t, and those items are exactly the ones maximizing her utility. Now assume that $\pi(t) = 0$. By Lemma 2(b), the empty set is feasible for t. Furthermore, for any item $s \in S$ the utility $u_t(s)$ is negative unless s is legal for t, in which case $u_t(s) = 0$. Notice that a buyer may decide to take or not to take any item with zero utility value. However, she gets a feasible set in both cases by the above, thus concluding the proof. ☐

4.2 Multi-demand markets up to three buyers

The aim of the section is to settle the existence of optimal dynamic prices in multi-demand markets with a bounded number of buyers. In order to present the results as clearly as possible, we follow the structure used in [3]: we first consider the case when (OPT) is satisfied, then extend the proof to the general setting in the Appendix.

Theorem 8 (Berger et al.). *Every multi-demand market with at most three buyers admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

Proof. We prove the theorem for instances where (OPT) is satisfied; the proof for the general case is detailed in the Appendix. By Lemma 3, it suffices to show the existence of an adequate ordering for G_π, where π is a weighted cover provided by Lemma 2. For a single buyer, the statement is meaningless. For two buyers t_1 and t_2, $|S| = b(t_1) + b(t_2)$ by assumption (OPT).

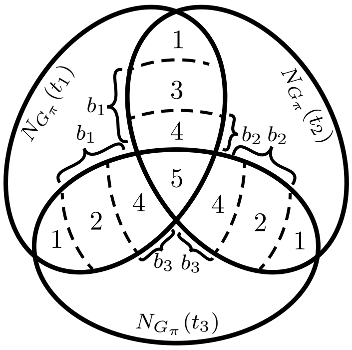


Figure 2: Definition of the labeling Θ for three buyers. Notice that some parts might be empty, e.g. if $|X_{12}| \leq b_2$, then there are no items with label 1 or 3 in the intersection of $N_{G_\pi}(t_1)$ and $N_{G_\pi}(t_2)$.

Let σ be an ordering that starts with items in $N_{G_\pi}(t_1) \triangle N_{G_\pi}(t_2)$ and then puts the items in $N_{G_\pi}(t_1) \cap N_{G_\pi}(t_2)$ at the end of the ordering. Then, after the deletion of the first $b(t_i)$ neighbors of t_i according to σ, the remaining $b(t_{3-i})$ items are in $N_{G_\pi}(t_{3-i})$, hence σ is adequate.

Now we turn to the case of three buyers. Let t_1, t_2 and t_3 denote the buyers, and let b_i, v_i, and u_i denote the demand, valuation, and utility function corresponding to buyer t_i, respectively. Without loss of generality, we may assume that $b_1 \geq b_2 \geq b_3$. The proof is based on the observation that a set is feasible if and only if its deletion leaves 'enough' items for the remaining buyers, formalized as follows.

Claim 1. *A set $F \subseteq N_{G_\pi}(t_i)$ is feasible for t_i if and only if $|F| = b_i$ and $|N_{G_\pi}(t_j) - F| \geq b_j$ for $j \neq i$.*

Proof. The conditions are clearly necessary. To prove sufficiency, we show that the constraints of Theorem 6(a) are fulfilled after deleting t_i and F from G_π, that is, $|S - F| = b(T) - b_i$ and $|N_{G_\pi}(Y) - F| \geq b(Y)$ for $Y \subseteq T - t_i$. By (OPT) and the assumption that every item is legal for at least two buyers, $|S - F| = b(T) - b_i$ holds for $Y = T - t_i$. Furthermore, one-element subsets have enough neighbors by assumption, and the claim follows. $\square$

For $I \subseteq \{1, 2, 3\}$, let $X_I \subseteq S$ denote the set of items that are legal exactly for buyers with indices in I, that is, $X_I := \left(\bigcap_{i \in I} N_{G_\pi}(t_i)\right) - \left(\bigcup_{i \notin I} N_{G_\pi}(t_i)\right)$. We may assume that $X_1 = X_2 = X_3 = \emptyset$. Indeed, given an adequate ordering for $G_\pi - (X_1 \cup X_2 \cup X_3)$ where the demands of t_i is changed to $b_i - |X_i|$ for $i \in \{1, 2, 3\}$, putting the items in $X_1 \cup X_2 \cup X_3$ at the beginning of the ordering results in an adequate solution for the original instance.

By assumption, $|X_{12}| + |X_{13}| + |X_{23}| + |X_{123}| = b_1 + b_2 + b_3$. Furthermore, $|X_{ij}| \leq b_i + b_j$ holds for $i \neq j$, as otherwise in any allocation there exists an item that is legal only for t_i and t_j but is not allocated to any of them, contradicting (OPT). We first define a labeling $\Theta : S \to \{1, 2, 3, 4, 5\}$ so that for each buyer i and set X_{ij}, the number of items in X_{ij} with label at most $4 - i$ is $\max\{0, |X_{ij}| - b_j\}$. We will make sure that each buyer i selects all items with label at most $4 - i$, which will be the key to satisfy the constraints of Claim 1, see Figure 2.

All the items in X_{123} are labeled by 5. If $|X_{12}| \leq b_2$, then all the items in X_{12} are labeled by 4. If $b_1 \geq |X_{12}| > b_2$, then b_2 items are labeled by 4 and the remaining $|X_{12}| - b_2$ items are labeled by 3 in X_{12}. If $|X_{12}| > b_1$, b_2 items are labeled by 4, $b_1 - b_2$ items are labeled by 3, and the remaining $|X_{12}| - b_1$ items are labeled by 1 in X_{12}. We proceed with X_{13} analogously. If $|X_{13}| \leq b_3$, then all the items in X_{13} are labeled by 4. If $b_1 \geq |X_{13}| > b_3$, then b_3 items are labeled by 4 and the remaining $|X_{13}| - b_3$ items are labeled by 2 in X_{13}. If $|X_{13}| > b_1$, b_3 items are labeled by 4, $b_1 - b_3$ items are labeled by 2, and the remaining $|X_{13}| - b_1$ items are labeled by 1 in X_{13}. Similarly, if $|X_{23}| \leq b_3$, then all the items in X_{23} are labeled by 4. If $b_2 \geq |X_{23}| > b_3$, then b_3 items are labeled by 4 and the remaining $|X_{23}| - b_3$ items are labeled by 2 in X_{23}. If $|X_{23}| > b_2$, then b_3 items are labeled by 4, $b_2 - b_3$ items are labeled by 2, and the remaining $|X_{23}| - b_2$ items are labeled by 1 in X_{23}.

Now let σ be any ordering of the items satisfying the following condition: if the label of item s_1 is strictly less than that of item s_2, then s_1 precedes s_2 in the ordering, that is, $\Theta(s_1) < \Theta(s_2)$ implies $\sigma(s_1) < \sigma(s_2)$. We claim that σ is adequate for G_π. To see this, it suffices to verify that the set F of the first $b(t_i)$ neighbors of t_i according to σ fulfills the requirements of Claim 1 for $i = 1, 2, 3$. Let $\{i, j, k\} = \{1, 2, 3\}$. First we show that F contains all the items $s \in X_{ij} \cup X_{ik}$ with $\Theta(s) \leq 4 - i$.

Claim 2. *We have* $|\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4 - i\}| \leq b_i$.

Proof. Suppose to the contrary that this does not hold. Then $b_i < \max\{0, |X_{ij}| - b_j\} + \max\{0, |X_{ik}| - b_k\}$ by the definition of the labeling. Since $|X_{ij}| \leq b_i + b_j$ and $|X_{ik}| \leq b_i + b_k$, we have $\max\{0, |X_{ij}| - b_j\} \leq b_i$ and $\max\{0, |X_{ik}| - b_k\} \leq b_i$. Therefore if $b_i < \max\{0, |X_{ij}| - b_j\} + \max\{0, |X_{ik}| - b_k\}$, then both maximums must be positive on the right hand side. However, this leads to $b_i + b_j + b_k < |X_{ij}| + |X_{ik}|$, contradicting $b_i + b_j + b_k = |X_{ij}| + |X_{ik}| + |X_{jk}| + |X_{ijk}|$. $\square$

By Claim 2, F contains all the items $s \in X_{ij} \cup X_{ik}$ with $\Theta(s) \leq 4 - i$, we have $|X_{ij} - F| \leq b_j$ and $|X_{ik} - F| \leq b_k$. Thus we get

$$\begin{aligned}
|N_{G_\pi}(t_j) - F| &= |X_{ij} - F| + |X_{jk}| + |X_{ijk} - F| \\
&= |S| - |X_{ik} - F| - |F| \\
&\geq (b_i + b_j + b_k) - b_k - b_i \\
&= b_j.
\end{aligned}$$

An analogous computation shows that $|N_{G_\pi}(t_k) - F| \geq b_k$. That is, F is indeed a feasible set for t_i, concluding the proof of the theorem. $\square$

5 Bi-demand markets

This section is devoted to the proof of our main result, the existence of optimal dynamic prices in bi-demand markets. The algorithms tries to identify subsets of buyers whose neighboring set in G_π is 'small', meaning that other buyers should take no or at most one item from it. If no such set exists, then an adequate ordering is easy to find. Otherwise, by examining the structure of dangerous sets, the problem is reduced to smaller instances.

Theorem 3. *Every bi-demand market admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

Proof. We prove the theorem for instances where (OPT) is satisfied; the proof for the general case is detailed in the Appendix. Let $G = (S, T; E)$ and w be the bipartite graph and weight function associated with the market. Take a weighted cover π of w provided by Lemma 2, and consider the subgraph $G_\pi = (S, T; E_\pi)$ of tight edges. For simplicity, we call a subset $M \subseteq E_\pi$ a $(1, 2)$-*factor* if $d_M(s) = 1$ for every $s \in S$ and $d_M(t) = 2$ for every $t \in T$. By (OPT), Lemma 1, and the assumption that all items are allocated in every optimal allocation, there is a one-to-one correspondence between optimal allocations and $(1, 2)$-factors of G_π. Therefore, by Lemma 3, it suffices to show the existence of an adequate ordering σ for G_π.

We prove by induction on $|T|$. The statement clearly holds when $|T| = 1$, hence we assume that $|T| \geq 2$. As there exists such a solution by assumption, $|N_{G_\pi}(Y)| \geq 2|Y|$ for every $Y \subseteq T$ by Theorem 6(a). We distinguish three cases.

Case 1. $|N_{G_\pi}(Y)| \geq 2|Y| + 2$ for every $\emptyset \neq Y \subsetneq T$.

For any $t \in T$ and $s_1, s_2 \in N_{G_\pi}(t)$, the graph $G_\pi - \{s_1, s_2, t\}$ still satisfies the conditions of Theorem 6(a), hence $\{s_1, s_2\}$ is feasible for t. Therefore σ can be chosen arbitrarily.

Case 2. $|N_{G_\pi}(Y)| \geq 2|Y| + 1$ for $\emptyset \neq Y \subsetneq T$ and there exists Y for which equality holds.

We call a set $Y \subseteq T$ *dangerous* if $|N_{G_\pi}(Y)| = 2|Y| + 1$. By Theorem 6(a), a pair $\{s_1, s_2\} \subseteq N_{G_\pi}(t)$ is not feasible for buyer t if and only if there exists a dangerous set $Y \subseteq T - t$ with $s_1, s_2 \in N_{G_\pi}(Y)$. In such case we say that Y *belongs to buyer* t. Notice that the same dangerous set might belong to several buyers.

Claim 3. *Assume that Y_1 and Y_2 are dangerous sets with $Y_1 \cup Y_2 \subsetneq T$.*

(a) If $Y_1 \cap Y_2 = \emptyset$ and $N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2) \neq \emptyset$, then $|N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| = 1$ and $Y_1 \cup Y_2$ is dangerous.

(b) If $Y_1 \cap Y_2 \neq \emptyset$, then both $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$ are dangerous.

Proof. Observe that

$$\begin{aligned}
(2|Y_1| + 1) + (2|Y_2| + 1) &= |N_{G_\pi}(Y_1)| + |N_{G_\pi}(Y_2)| \\
&= |N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| + |N_{G_\pi}(Y_1) \cup N_{G_\pi}(Y_2)| \\
&= |N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| + |N_{G_\pi}(Y_1 \cup Y_2)|.
\end{aligned}$$

Assume first that $Y_1 \cap Y_2 = \emptyset$. Then $|N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| \leq 1$ as otherwise $|N_{G_\pi}(Y_1 \cup Y_2)| \leq 2(|Y_1| + |Y_2|) = 2|Y_1 \cup Y_2|$, contradicting the assumption of Case 2. If $|N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| = 1$, then $|N_{G_\pi}(Y_1 \cup Y_2)| = 2|Y_1 \cup Y_2| + 1$ and so $Y_1 \cup Y_2$ is dangerous.

Now consider the case when $Y_1 \cap Y_2 \neq \emptyset$. Then

$$\begin{aligned}
|N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| + |N_{G_\pi}(Y_1 \cup Y_2)| &\geq |N_{G_\pi}(Y_1 \cap Y_2)| + |N_{G_\pi}(Y_1 \cup Y_2)| \\
&\geq (2|Y_1 \cap Y_2| + 1) + (2|Y_1 \cup Y_2| + 1) \\
&= (2|Y_1| + 1) + (2|Y_2| + 1).
\end{aligned}$$

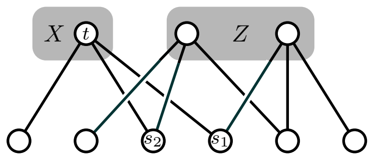
Therefore we have equality throughout, implying that both $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$ are dangerous. $\square$

Let Z be an inclusionwise maximal dangerous set.

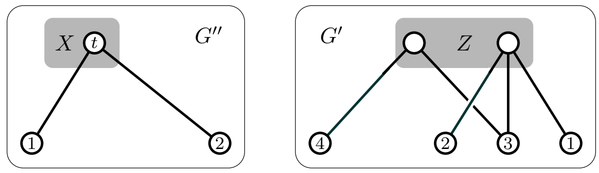
Subcase 2.1. There is no dangerous set disjoint from Z.

First we show that if a pair $s_1, s_2 \in N_{G_\pi}(t)$ is not feasible for a buyer $t \in T - Z$, then $s_1, s_2 \in N_{G_\pi}(Z)$. Indeed, if $\{s_1, s_2\}$ is not feasible for t, then there is a dangerous set X belonging to t with $s_1, s_2 \in N_{G_\pi}(X)$. Since $t \notin X \cup Z$ and $Z \cap X \neq \emptyset$ by the assumption of the subcase, Claim 3(b) can be applied and we get that $X \cup Z$ is dangerous as well. The maximal choice of Z implies $X \cup Z = Z$, hence Z belongs to t and $s_1, s_2 \in N_{G_\pi}(Z)$.

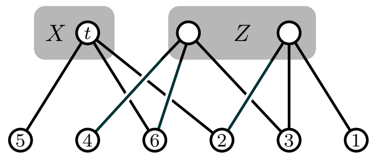
Now take an arbitrary buyer $t_0 \in T - Z$ who shares a neighbor with Z and let $s_0 \in N_{G_\pi}(t) \cap N_{G_\pi}(Z)$. Let σ' be an arbitrary ordering of the items in $S - N_{G_\pi}(Z)$. Furthermore, Let G'' be the graph obtained by deleting the items in $S - (N_{G_\pi}(Z) - s_0)$ and the buyers in $T - Z$. As every edge is contained in a $(1, 2)$-factor, G'' admits a $(1, 2)$-factor as well. By induction, there exists an adequate ordering σ'' of the items in G''. Finally, let σ''' denote the trivial ordering of the single element set $\{s_0\}$. Let $\sigma := (\sigma', \sigma'', \sigma''')$. Then any buyer $t \in T - Z$ will choose at most one item from $N_{G_\pi}(Z)$, hence the adequateness of σ follows from that of σ'' and the assumption of the subcase.



(a) The graph of tight edges corresponding to the instance on Figure 1, where Z is an inclusionwise maximal dangerous set, and X is an inclusionwise minimal dangerous set disjoint from Z.



(b) The graphs $G' = G[Z \cup N_{G_\pi}(Z) - s_2]$ and $G'' = G[X \cup N_{G_\pi}(X) - s_2]$, together with adequate orderings σ' and σ'', respectively.



(c) Construction of the ordering $\sigma = (\sigma', \sigma''|_{N_{G_\pi}(X)-s_2}, \sigma''')$, where σ''' is the trivial ordering of the one element set $\{s_2\}$.

Figure 3: An illustration of the inductive step in Subcase 2.2.2.

Subcase 2.2. There exists a dangerous set disjoint from Z.

Let X be an inclusionwise minimal dangerous set disjoint from Z.

Subcase 2.2.1. For any $t \in X$ and for any $s_1, s_2 \in N_{G_\pi}(t)$, the set $\{s_1, s_2\}$ is feasible.

Take an item $s_0 \in N_{G_\pi}(X)$ that has a neighbor $t_0 \in T - X$. Let G' denote the graph obtained by deleting X and $N_{G_\pi}(X) - s_0$. It is not difficult to check that G' admits a $(1,2)$-factor as well. By induction, there exists an adequate ordering σ' of the items in G'. Let σ'' be an arbitrary ordering of the items in $N_{G_\pi}(X) - s_0$, and define $\sigma := (\sigma', \sigma'')$. Then t_0 chooses at most one item from $N_{G_\pi}(X)$ (namely s_0) as she has at least one neighbor outside of $N_{G_\pi}(X)$ and those items have smaller indices in the ordering. Thus the adequateness of σ follows from that of σ' and from the assumption that any pair $s_1, s_2 \in N_{G_\pi}(t)$ form a feasible set for $t \in X$.

Subcase 2.2.2. There exists $t \in X$ and $s_1, s_2 \in N_{G_\pi}(t)$ such that $\{s_1, s_2\}$ is not feasible.

The following claim is the key observation of the proof.

Claim 4. *$X \cup Z = T$ and $N_{G_\pi}(X) \cap N_{G_\pi}(Z) = \{s_1, s_2\}$.*

Proof. Let $Y \subseteq T - t$ be a dangerous set with $s_1, s_2 \in N_{G_\pi}(t)$. As $t \in T - (Z \cup Y)$ and Z is inclusionwise maximal, either $Y \subseteq Z$ or $Y \cap Z = \emptyset$ by Claim 3(b). In the latter case, X and Y are dangerous sets with $X \cup Y \subsetneq T$. Furthermore, $|N_{G_\pi}(X) \cap N_{G_\pi}(Y)| \geq 2$ since s_1 and s_2 are contained in both. Hence, by Claim 3(a), $X \cap Y \neq \emptyset$. But then $X \cap Y$ is dangerous by Claim 3(b), contradicting the minimality of X. Therefore $Y \subseteq Z$. By Claim 3(a), $X \cup Z = T$. As $|N_{G_\pi}(X)| = 2|X|+1$, $|N_{G_\pi}(Z) = 2|Z|+1$, and $|S| = 2|T| = 2|T|+2|Z|$, the claim follows. □

Let G' and G'' denote the graphs obtained by deleting $X \cup (N_{G_\pi}(X) - s_2)$ and $Z \cup N_{G_\pi}(Z) - s_2$, respectively. As every edge is contained in a $(1,2)$-factor, both G' and G'' admit a $(1,2)$-factor. By induction, there exists adequate orderings σ' and σ'' of the items in G' and G'', respectively. Finally, let σ''' denote the trivial ordering of the single element set $\{s_2\}$. Let $\sigma := (\sigma', \sigma''|_{N_{G_\pi}(X)-s_1}, \sigma''')$. We claim that σ is adequate. Indeed, if a buyer $t \in Z$ arrives first, then she chooses two items from $N_{G_\pi}(Z) - s_2$ according to σ'. As σ' is adequate for G' and

$G'' - s_1 + s_2$ has a $(1,2)$-factor, the remaining graph has a $(1,2)$-factor as well. If a buyer $t \in X$ arrives first, then she chooses two items from $N_{G_\pi}(X) - s_2$. By Claim 4, these items form a feasible set.

Case 3. $|N_{G_\pi}(T')| = 2|T'|$ for some $\emptyset \neq T' \subsetneq T$.

We claim that there exists a set T' satisfying the assumption if and only if G_π is not connected. Indeed, if G_π is not connected, then necessarily the number of items is exactly twice the number of buyers in every component as the graph is supposed to have a $(1,2)$-factor. To see the other direction, let $S' := N_{G_\pi}(T')$, $T'' := T - T'$, $S'' := S - S'$, and consider the subgraphs $G' := G_\pi[T' \cup S']$ and $G'' := G_\pi[T'' \cup S'']$. As every tight edge is legal and all the vertices in S' are matched to vertices in T' in any optimum b-matching, G_π contains no edges between T'' and S'. Therefore G_π is not connected, and it is the union of G' and G''. By induction, there exist adequate orderings σ' and σ'' of S' and S'', respectively. Then the ordering $\sigma := (\sigma', \sigma'')$ is adequate with respect to π.

By Lemma 2, π can be determined in polynomial time. Then, the inductive proof provides a polynomial time algorithm for determining an adequate ordering for G_π. To see this, it remains to show that one can find an inclusionwise maximal or minimal dangerous set, if exists, in G_π. This can be done as follows: take two copies of each vertex $t \in T$, and connecting them to the vertices in $N_{G_\pi}(t)$. Furthermore, add a dummy vertex w_0 to the graph and connect it to every vertex in S. Let $G' = (S', T'; E')$ denote the graph thus obtained. For a set $Y \subseteq T$, let $Y' \subseteq T'$ consist of the copies of the vertices in Y plus the vertex w_0. It is not difficult to check that $Y \subseteq T$ is an inclusionwise minimal or maximal dangerous set of G_π if and only if Y' is an inclusionwise minimal or maximal subset of T' with $|N_{G'}(Y')| = |Y'|$. Hence Y can be determined by relying on Kőnig's alternating path algorithm [21]. $\square$

Remark 9. Theorem 3 settles the existence of optimal dynamic prices when the demand of each buyer is exactly two. However, the proof can be straightforwardly extended to the case when the demand of each buyer is at most two.

6 Conclusions and open problems

This work focuses on the existence of optimal dynamic prices for multi-demand valuations. By relying on structural properties of an optimal dual solution, we gave polynomial-time algorithms for determining such prices for unit-demand markets and for multi-demand markets up to three buyers, thus giving new interpretations of results of Cohen-Addad et al. and Berger et al. We also proved that any bi-demand market has a dynamic pricing scheme that achieves optimal social welfare. An open problem is to decide the existence of optimal dynamic prices in multi-demand markets in general.

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Appendix

Our goal is to give optimal dynamic pricing schemes for multi-demand markets with at most three buyers and for bi-demand markets, without assuming (OPT). In both cases, the proof is based on the following idea: We add small valued dummy items to the market so that (OPT) is satisfied, then we determine optimal dynamic prices for the modified market, and show that the same prices are optimal for the original instance as well.

Formally, consider a market for which (OPT) does not hold, that is, the number of items is less than the total demand of the buyers. Let $G = (S, T; E)$ be the bipartite graph associated with the market, and take a minimum weighted covering π provided by Lemma 2. For ease of discussion, let us denote the set of buyers who might receive less items than their demand in an optimal solution by

$$\hat{T} := \{t \in T \mid d_M(t) < b(t) \text{ for some maximum weight } b\text{-matching } M\}.$$

We call the items in S *real*. Now extend the graph by adding a set $\hat{S}$ of $b(T) - |S|$ *dummy items*; we refer to edges going between these items and buyers as *dummy edges*. We define the value of a dummy item (and so the weight of the corresponding dummy edge) to be 2ε for each buyer, where $\varepsilon := 1/4 \cdot \Delta(\pi)$. By Lemma 2(b) and the assumption that every item is used in every optimal allocation, ε is strictly positive, hence the modified instance satisfies (OPT). Let $G^+ = (S^+, T, E^+)$ and w^+ denote the graph and weight function thus obtained, respectively. It is not difficult to check that the maximum weight b-factors of G^+ are exactly those that can be obtained from a maximum weight b-matching of G by adding $|\hat{S}|$ dummy edges.

Lemma 4. *There exists a minimum weighted covering π^+ of w^+ such that* *(a)* $\pi^+(t) = \varepsilon$ *for each* $t \in \hat{T}$*,*

(b) $\pi^+(t) > \varepsilon$ *for each* $t \in T - \hat{T}$*, and*

(c) if $t \in \hat{T}$*,* $s \in S$ *and* st *is not legal, then* $w(st) - \pi^+(s) < 0$*.*

Furthermore, such a π^+ *can be determined in polynomial time.*

Proof. Let π^+ be an extension of π by setting $\pi^+(\hat{s}) := 2\varepsilon$ for $\hat{s} \in \hat{S}$. It is not difficult to check that π^+ is a weighted covering of w^+. Furthermore, as the total value of π^+ equals the total value of π plus $2\varepsilon|\hat{S}|$ which is exactly the difference between the maximum weight of a b-factor in G^+ and the maximum weight of a b-matching in G, π^+ is a minimum weighted covering.

Now increase $\pi^+(t)$ by ε for $t \in T$ and decrease $\pi^+(s)$ by ε for $s \in S^+$. As (OPT) holds for the modified instance, the total value of π^+ does not change, hence it remains a minimum weighted covering. By Lemma 2(b), $\pi(t) = 0$ for $t \in \hat{T}$ and $\pi(t) > 0$ otherwise. Furthermore, by the assumption that every item is used in every optimal allocation, $\pi(s) > 0$ for $s \in S$. These together show that π^+ satisfies (a) and (b).

By Lemma 2(a), for every $st \in E$ such that $t \in \hat{T}$, $s \in S$, and st is not legal, we have $w(st) - \pi(s) < \pi(t) = 0$, therefore $w(st) - \pi^+(s) < 0$ by the choice of ϵ. This proves the last part of the claim. $\square$

A. Multi-demand markets up to three buyers

Theorem 2 (Berger et al.)**.** *Every multi-demand market up to three buyers admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

Proof. For a single buyer, the statement is meaningless.

For two buyers t_1 and t_2, if the dummy items are in $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$, labelling items in $N_{G^+_{\pi^+}}(t_1) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$ and $N_{G^+_{\pi^+}}(t_2) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$ by 1 and items in $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$ by 2 results in optimal allocations, because for $i = 1, 2$, buyer t_i has positive utility for all real items in $N_{G^+_{\pi^+}}(t_i)$, negative utility for items not in $N_{G^+_{\pi^+}}(t_i)$, and she prefers items in $N_{G^+_{\pi^+}}(t_i) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$. If the dummy items are in, say, $N_{G^+_{\pi^+}}(t_1) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$, we chose $\max\{0, b_2 - |N_{G^+_{\pi^+}}(t_2) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))|\}$ items from $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$ and increase their prices by ε. This way, t_2 gets all items which are legal only for her and she gets $\max\{0, b_2 - |N_{G^+_{\pi^+}}(t_2) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))|\}$ items from $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$ as her utility is still positive for them. Buyer t_1 takes real items in $N_{G^+_{\pi^+}}(t_1) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$ and the items in $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$ whose prices remained unchanged.

Now we turn to the case of three buyers. Add dummy items to the instance as described before, and let π^+ be a minimum weighted covering provided by Lemma 4. Let t_1, t_2 and t_3 denote the buyers, and let b_i, v_i, and u_i denote the demand, valuation, and utility function corresponding to buyer t_i, respectively. For $I \subseteq \{1, 2, 3\}$, let $X_I \subseteq S^+$ denote the set of items that are legal exactly for buyers with indices in I, that is, $X_I := \left(\bigcap_{i \in I} N_{G^+_{\pi^+}}(t_i)\right) - \left(\bigcup_{i \notin I} N_{G^+_{\pi^+}}(t_i)\right)$. Without loss of generality, we may assume that $b_1 \geq b_2 \geq b_3$. However, unlike before, we cannot assume $X_1 = X_2 = X_3 = \emptyset$ due to the presence of dummy items.

Similarly to the case when property (OPT) holds, we define a labeling $\Theta : S^+ \to \{1, 2, 3, 4, 5\}$ such that any b_i items with the smallest labels in $N_{G^+_{\pi^+}}(t_i)$ form a feasible set for t_i. That is, for an appropriately small $\delta > 0$, setting the prices to $\pi^+(s) + \delta \cdot \Theta(s)$ for each item s where $\delta := \Delta(\pi^+)/(|S|+1)$, results in optimal dynamic prices for the modified instance. Unfortunately, when the prices are restricted to the set of original items, optimality might not met due to the absence of dummy items. This is because a buyer might replace the missing dummy items by real items that she did not take before, which results in a suboptimal solution. To resolve this, as in the bi-demand case, we further increase the prices by ε to ensure that buyers have negative utility from items they should not choose. Notice that Observation 10 holds again.

We have seen, when the market satisfy the property (OPT) and $X_1 = X_2 = X_3 = \emptyset$, it is enough to ensure $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\} \subseteq F$ for $\{i,j,k\} = \{1,2,3\}$, where F be a set of b_i items with the largest utility for t_i. Now, if there are $\hat{s}$ dummy items with label $\Theta(\hat{s}) \leq 4-i$, t_i simply skips them, so we also have to ensure she does not take too much real items with label greater than $4-i$. If some X_i ($i \in \{1,2,3\}$) is not empty, but it contains only real items, if we label them by 0, t_i always buys them, therefore we can reduce the problem to the case when X_i is empty and the demand of t_i is $b_i - |X_i|$. If some X_i contains dummy items, the reduction will be more difficult. The following claim shows how the conditions for the feasible sets change when $X_1 = X_2 = X_3 = \emptyset$:

Claim 5. *Assume $X_1 = X_2 = X_3 = \emptyset$. Let $i \in \{1,2,3\}$ and let F be the following set: if t_i has at positive utility for at least b_i items in $N_{G^+_{\pi^+}}(t_i)$, F is the set of the first b_i items with the largest utility. If t_i has at positive utility for less than b_i items in $N_{G^+_{\pi^+}}(t_i)$, F contains all of them. If*

(a) F contains all real items in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\}$,

(b) The difference $b_i - |F|$ is at least the number of dummy items in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\}$,

(c) The difference $b_i - |F|$ is at most the number of dummy items in $N_{G^+_{\pi^+}}(t_i)$,

then F is feasible for t_i.

Proof. Let F be a set of items in $N_{G^+_{\pi^+}}(t_i)$ as stated above. We extend F with dummy items the following way: If there are dummy items in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\}$, we add them to F. If the cardinality of the set we got this way is strictly less than b_i, we further extend it by adding dummy items from $N_{G^+_{\pi^+}}(t_i)$ with label at least 4 until the resulting sets cardinality becomes b_i. By 5(b) and 5(c), this can be achieved. Let F' denote the resulting set. Then $|F'| = b_i$ and F' contains all real and dummy items in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\}$. We have $|X_{ij} - F'| \leq b_j$ and $|X_{ik} - F'| \leq b_k$. Thus we get

$$\begin{aligned}
|N_{G^+_{\pi^+}}(t_j) - F'| &= |X_{ij} - F'| + |X_{jk}| + |X_{ijk} - F'| \\
&= |S| - |X_{ik} - F'| - |F'| \\
&\geq (b_i + b_j + b_k) - b_k - b_i \\
&= b_j.
\end{aligned}$$

An analogous computation shows that $|N_{G^+_{\pi^+}}(t_k) - F'| \geq b_k$. Since we get F' by adding only dummy items to F, this proves the feasibility of F. $\square$

We will apply a similar labeling procedure as when property OPT holds, then increase some prices by ε. As dummy items are completely equivalent, either none or all of them are legal for each buyer. We divide the proof into three cases based on whether dummy items are legal only for two or all three of the buyers.

Case 1. The dummy items are in X_i for some $i \in \{1,2,3\}$.

By Lemma 4, each buyer has positive utility from her legal real items and negative utility from her non-legal items.

If the number of dummy items is b_i, then all real items are non-legal for t_i, and her utility from real items is negative. Therefore we can apply the labeling procedure for the other two buyers. Otherwise, let b_i' denote the difference between b_i and the number of dummy items, that is, $b_i' := b_i - |\hat{S}|$. If there are real items in X_i, we label them by 0 and decrease b_i' by the number of real items in X_i. Regardless if there are real items in X_i or not, $b_i' = b_i - |X_i|$. We delete the dummy items from the graph and the real items from X_i, if there is any, then apply the labeling procedure for three buyers, but with b_i' in place of b_i. Now $|\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \le 4-i\}| \le b_i'$. We select $\max(0, b_i' - |\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \le 4-i\}|)$ items from $X_{ijk} \cup \{s \in X_{ij} \cup X_{ik} \mid \Theta(s) > 4-i\}$, starting with the items with lower labels, and leave their prices unchanged, while the prices of all other items in $X_{ijk} \cup \{s \in X_{ij} \cup X_{ik} \mid \Theta(s) > 4-i\}$ are increased by ε. This way we achieve that t_i has non-negative utility from exactly b_i' items. Despite the price increasing, t_j $(j \ne i)$ has positive utility for all items in $N_{G^+_{\pi^+}}(t_j)$ and since we start the price increasing with the items with lower labels, the order of items in $N_{G^+_{\pi^+}}(t_j)$ does not change, and the conditions of Claim 5 hold.

From now on, we can assume $X_1 = X_2 = X_3 = \emptyset$. Otherwise, we label the items in X_i by 0, delete them from the graph, and replace b_i by $b_i - |X_i|$.

Case 2. The dummy items are in X_{13}.

We apply a similar labeling procedure that we used when the market satisfies property (OPT). The items in X_{123} get label 5. As before, items in X_{ij} are labeled by 4, θ or 1, where $\theta = 3$ if $\{i,j\} = \{1,2\}$, otherwise $\theta = 2$. However, dummy items are preferred to get higher labels. That is, we label as many dummy items by 4 as possible, and if the number of dummy items is more than the number of items to be labeled by 4, we proceed with labeling dummy items by 2, and then by 1 if necessary. The proof of Theorem 8 shows that t_i takes every item in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \le 4-i\}$.

We distinguish three subcases:

Subcase 1. $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \le 2\}| > b_3$ and there is no item in X_{13} with label 1.

We do not change the prices in X_{23}. If $b_1 \le |X_{13}| + |\{s \in X_{12} \mid \Theta(s) \le 3\}|$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ by ε, otherwise we select $b_1 - (|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \le 3\})$ items from $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$, starting with the ones in X_{12}. We leave the prices of the selected items unchanged, but we increase the prices of items in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ which were not selected by ε. This way, t_3 takes all items in X_{23} with label 1 (remember, there are no items in X_{13} with label 1) as they are real items. t_2 takes all items in X_{23} with label 1 and 2, since their prices were not increased. If $b_1 \le |X_{13}| + |\{s \in X_{12} \mid \Theta(s) \le 3\}|$, t_1 takes all items from $\{s \in X_{12} \mid \Theta(s) \le 3\}$ as these are real items, and t_1 gets all real or dummy items in $\{s \in X_{12} \mid \Theta(s) \le 2\}$, since we increased the prices in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$. If $b_1 > |X_{13}| + |\{s \in X_{12} \mid \Theta(s) \le 3\}|$, t_1 gets X_{13} and all items in $X_{12} \cup X_{123}$ with unchanged prices. In both cases, t_1 takes all real items in X_{12} with label 1 and 3, and she also takes all real items in X_{13} with label 2. Moreover, the difference between b_1 and the real items she takes is at least the number of dummy items in X_{13} with label 1 and 2. The way we increased some prices, we ensured conditions 5(b), 5(c) are fulfilled.

Subcase 2. $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \le 2\}| > b_3$ and exists an item in X_{13} with label 1.

As in the previous case, we need to ensure 5(b), 5(c) hold. In this case, $|\{s \in X_{13} \mid \Theta(s) > 1\}| = b_1$ and $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \le 2\}| > b_3$, which implies

$|X_{12}| + |X_{123}| + |\{s \in X_{23} \mid \Theta(s) = 4\}| = b_2$, therefore there is no item in X_{12} with label 1 or 3. We increase the prices in $X_{12} \cup X_{123}$ by ε. We select $b_3 - |\{s \in X_{13} \mid \Theta(s) = 1\}| - |\{s \in X_{23} \mid \Theta(s) = 1\}|$ items from $\{s \in X_{23} \mid \Theta(s) = 2\}$ (the assumption $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| > b_3$ shows this can be done), and leave their prices unchanged, but we increase the prices of the remaining items in $\{s \in X_{23} \mid \Theta(s) = 2\}$ by ε, and we also increase the prices in $\{s \in X_{23} \mid \Theta(s) = 4\}$ by ε. This way, t_1 only takes items from X_{13}, which is enough, since there are no items in X_{12} with label 1 or 3. t_2 takes all items in X_{23} with label 1 and 2, and t_3 takes all real items in $X_{13} \cup X_{23}$ with label 1.

Subcase 3. $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| \leq b_3$.

In the case when property (OPT) holds, t_1 could choose freely from $\{s \in X_{12} \cup X_{13} \mid \Theta(s) = 4\}$ when $|\{s \in X_{12} \cup X_{13} \mid \Theta(s) \leq 3\}| < b_1$, now we will force her to buy as many items from $\{s \in X_{13} \mid \Theta(s) = 4\}$ as possible. We do this in the following way: if $|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \geq b_1$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ by ε. If $|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| < b_1$, we choose $b_1 - (|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}|)$ items from $\{s \in X_{12} \mid \Theta(s) = 4\}$, and if the items in $\{s \in X_{12} \mid \Theta(s) = 4\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ which were not chosen by ε. We do the same with t_3. If we have to choose items from X_{123}, we start with the items which are chosen because of t_1, if there is any. If there is no chosen item because of t_1 or we have to choose more, we choose from the items with increased price, but we decrease their price by ε. First, we check the case when the first buyer is t_1, and assume we increased the price of all items in X_{123}. If $|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \geq b_1$, then t_1 has negative utility for the items not in $X_{13} \cup \{s \in X_{12} \mid \Theta(s) \leq 3\}$. Since $|\{s \in X_{13} \cup X_{12} \mid \Theta(s) \leq 3\}| \leq b_1$, t_1 takes all real items in $\{s \in X_{13} \mid \Theta(s) \leq 3\}$ and she also takes real items in $\{s \in X_{13} \mid \Theta(s) \leq 2\}$. By the price increasing, 5(b) and 5(c) also hold. If $|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| < b_1$, t_1 gets all real items in $X_{13} \cup \{s \in X_{12} \mid \Theta(s) \leq 3\}$ and the items in $\{s \in X_{12} \mid \Theta(s) = 4\}$ whose price were not changed. One can verify that 5(b) and 5(c) hold again. If there are items in X_{123} with unchanged prices, and the number of them is $b_1 - |X_{12}| - |X_{13}|$, t_1 gets all real items in X_{13}, X_{12} and the items in X_{123} whose price were not changed. 5(b) and 5(c) hold again. The remaining case is when exists at least one item in X_{123} whose price was not changed and the number of items in X_{123} with unchanged prices is greater than $b_1 - |X_{12}| - |X_{13}|$. That means we left their prices unchanged because of t_3, that is $|X_{13}| + |X_{23}| < b_3$. That also means there is no item in X_{13} with label 1 or 2. It is not difficult to see that t_1 takes all items in $\{s \in X_{12} \mid \Theta(s) \leq 3\}$. Secondly, if the first buyer is t_3, the reasoning goes the same way as with t_1: if we increased the price of all items in X_{123} and $|X_{13}| + |\{s \in X_{23} \mid \Theta(s) = 1\}| \geq b_3$, then t_3 has negative utility for the items not in $X_{13} \cup \{s \in X_{23} \mid \Theta(s) = 1\}$. Since $|\{s \in X_{13} \cup X_{23} \mid \Theta(s) = 1\}| \leq b_3$, t_3 takes all real items in $\{s \in X_{13} \mid \Theta(s) = 1\}$ and she also gets $\{s \in X_{23} \mid \Theta(s) = 1\}$. If $|X_{13}| + |\{s \in X_{23} \mid \Theta(s) = 1\}| < b_3$, t_3 buys all real items in $X_{13} \cup \{s \in X_{23} \mid \Theta(s) = 1\}$ and the items in $\{s \in X_{23} \mid \Theta(s) > 1\}$ whose price were not changed. 5(b) and 5(c) holds again. If there are items in X_{123} with unchanged prices, and the number of them is $b_3 - |X_{13}| - |X_{23}|$, t_3 buys all real items in X_{13}, X_{23} and the items in X_{123} whose price were not changed. When exists at least one item in X_{123} whose price was not changed and the number of items in X_{123} with unchanged prices is greater than $b_3 - |X_{13}| - |X_{23}|$, $|X_{12}| + |X_{13}| < b_1$ holds. That means there is no item in X_{13} with label 1. It is not difficult to see that t_3 gets all items in $\{s \in X_{23} \mid \Theta(s) = 1\}$. It is not difficult to check 5(b) and 5(c) holds. Finally, if the first buyer is t_2, she gets all items in $\{s \in X_{23} \mid \Theta(s) \leq 2\} \cup \{s \in X_{12} \mid \Theta(s) = 1\}$ as we only increased prices in $\{s \in X_{12} \cup X_{23} \cup X_{123} \mid \Theta(s) \geq 4\}$.

Case 3. The dummy items are in X_{12}.

The initial labeling procedure is the same as in Case 2, then we increase some of the prices.

First, we want to ensure 5(b), 5(c) holds if t_1 is the first buyer. We do this the following way: if $|X_{12}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| \geq b_1$, we increase the prices in $\{s \in X_{13} \mid \Theta(s) = 4\} \cup X_{123}$ by ε. If $|X_{12}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| < b_1$, we choose $b_1 - (|X_{12}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}|)$ items from $\{s \in X_{13} \mid \Theta(s) = 4\}$, and if the items in $\{s \in X_{13} \mid \Theta(s) = 4\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{13} \mid \Theta(s) = 4\} \cup X_{123}$ which were not chosen by ε. We proceed similarly with t_2 instead of t_1 to ensure 5(b), 5(c) holds if she is the first buyer in the market. If $|X_{12}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| \geq b_2$, we increase the prices in $\{s \in X_{23} \mid \Theta(s) = 4\} \cup X_{123}$ by ε. If $|X_{12}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| < b_2$, we choose $b_2 - (|X_{12}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}|)$ items from $\{s \in X_{23} \mid \Theta(s) = 4\}$, and if the items in $\{s \in X_{23} \mid \Theta(s) = 4\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{23} \mid \Theta(s) = 4\} \cup X_{123}$ which were not chosen by ε. If we have to choose items from X_{123}, we start with the items which are chosen because of t_1, if there is any. If there is no chosen item because of t_1 or we have to choose more, we choose from the items with increased price, but we decrease their price by ε.

Now let us assume t_1 is the first buyer. We also assume first that we increased the price of all items in X_{123}. t_1 gets all items from $\{s \in X_{13} \mid \Theta(s) \leq 2\}$ and she also takes the real items in $\{s \in X_{12} \mid \Theta(s) \leq 3\}$. If $|X_{12}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| < b_1$, she buys all real items in X_{12}, $\{s \in X_{13} \mid \Theta(s) \leq 2\}$ and the items in $\{s \in X_{13} \mid \Theta(s) = 4\}$ whose price were not changed. If there are items in X_{123} with unchanged prices, and the number of them is $b_1 - |X_{12}| - |X_{13}|$, t_1 gets all items in X_{12}, X_{13} and the items in X_{123} whose price were not changed. The remaining case is when exists at least one item in X_{123} whose price was not changed and the number of items in X_{123} with unchanged prices is greater than $b_1 - |X_{12}| - |X_{13}|$. That means we left their prices unchanged because of t_2, that is $|X_{12}| + |X_{23}| < b_2$. That also means there is no item in X_{12} with label 1 or 3. It is not difficult to see that t_1 buys all items in $\{s \in X_{13} \mid \Theta(s) \leq 2\}$, as their prices are unchanged. If t_2 is the first buyer, the reasoning goes similarly. The only thing which is different from the previous case is when there exists at least one item in X_{123} with unchanged price, but the number of these items is greater than $b_2 - |X_{12}| - |X_{23}|$. Now, it means $b_1 > |X_{12}| + |X_{13}|$, which does not mean there are no items in X_{12} with label 1 or 3, it only means there are no items in X_{12} with label 1, but that is enough as t_2 has to buy all real items in X_{12} with label 1, if there is any, but she can leave real items in X_{12} with label 3 or 4. If the first buyer is t_3, she takes all items in $\{s \in X_{13} \cup X_{23} \mid \Theta(s) = 1\}$.

Case 4. The dummy items are in X_{23}.

We start with the same labeling procedure as in Case 2 and Case 3. We distinguish five subcases:

Subcase 1. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| > b_2$ and there is at least one item in X_{23} with label 2.

The assumption that there is at least one item in X_{23} with label 2 shows $|\{s \in X_{23} \mid \Theta(s) = 4\}| = b_3$, thus with $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| > b_2$, it implies $|X_{13}| < b_1$, therefore there is no item in X_{13} with label 1. We increase the prices in $X_{13} \cup X_{123}$ by ε. If $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) = 1\}| \geq b_2$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) = 3\} \cup \{s \in X_{12} \mid \Theta(s) = 4\}$ by ε. If $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) = 1\}| < b_2$, we select some items from $\{s \in X_{12} \mid \Theta(s) = 3\}$ such way that the number of the selected items are $b_2 - |\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) = 1\}|$, and leave their prices unchanged, but we increase the prices of the unselected items in X_{12} with label 3 and the prices of the label 4 items in X_{12} by ε. This way, if t_1 comes first, she gets $\{s \in X_{12} \mid \Theta(s) \leq 3\}$ and $\{s \in X_{13} \mid \Theta(s) = 2\}$ (remember, there are no items in X_{13} with label 1). If t_2 comes first, she takes $\{s \in X_{12} \mid \Theta(s) = 1\}$ and the real items in $\{s \in X_{23} \mid \Theta(s) \leq 2\}$. If t_3 comes first, she gets the real items in $\{s \in X_{23} \mid \Theta(s) = 1\}$. The price increasing shows 5(b) and 5(c) holds.

Subcase 2. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| > b_2$ and there are no items in X_{23} with label 2.

We leave all prices unchanged. If t_1 comes first, she takes $\{s \in X_{13} \mid \Theta(s) \leq 2\}$ and $\{s \in X_{12} \mid \Theta(s) \leq 3\}$. If t_2 or t_3 comes first, they get all real items in $\{s \in X_{12} \mid \Theta(s) = 1\}$ and $\{s \in X_{13} \mid \Theta(s) = 1\}$, respectively. As the dummy items are in $\{x \in X_{23} \mid \Theta(s) = 4\}$, 5(b), 5(c) holds automatically.

Subcase 3. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \leq b_2$, $|\{s \in X_{23} \mid \Theta(s) = 1\}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| > b_3$ and exists an item in X_{23} with label 1.

The assumption that there is at least one item in X_{23} with label 1 shows $|\{s \in X_{23} \mid \Theta(s) > 1\}| = b_2$, thus with $|\{s \in X_{23} \mid \Theta(s) = 1\}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| > b_3$, it implies $|X_{13}| < b_1$, therefore there is no item in X_{13} with label 1. We increase the prices in $X_{12} \cup X_{123}$ by ε. We also increase the prices in $\{s \in X_{13} \mid \Theta(s) > 1\}$.

If t_1 comes first, she takes $\{s \in X_{12} \mid \Theta(s) = 3\}$ (there are no items in X_{12} with label 1) and $\{s \in X_{13} \mid \Theta(s) \leq 2\}$, as in $N_{G_\pi}(t_1)$, we only left the prices unchanged in $\{s \in X_{13} \mid \Theta(s) = 1\}$, which means the order of items in $N_{G_\pi}(t_1)$ remained unchanged. If t_2 comes first, she gets the real items in $\{s \in X_{23} \mid \Theta(s) \leq 2\}$. If the first buyer is t_3, she takes $\{s \in X_{13} \cup X_{23} \mid \Theta(s) = 1\}$. Therefore all three conditions of Claim 5 hold again.

Subcase 4. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \leq b_2$, $|\{s \in X_{23} \mid \Theta(s) = 1\}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| > b_3$ and there is no item in X_{23} with label 1.

If $|X_{23}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \geq b_2$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ by ε. If $|X_{23}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| < b_2$, we select $b_2 - (|X_{23}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}|)$ items from $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$, starting with the items in $\{s \in X_{12} \mid \Theta(s) = 4\}$, and we increase the prices of the unselected items in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ by ε.

When the first buyer is t_1, she takes $\{s \in X_{12} \mid \Theta(s) \leq 3\} \cup \{s \in X_{13} \mid \Theta(s) \leq 2\}$, as we only increased prices of items with label 4 or 5. If the first buyer is t_2, she gets $\{s \in X_{23} \mid \Theta(s) = 2\}$ (remember, there is no item in X_{23} with label 1) and she also buys $\{s \in X_{12} \mid \Theta(s) \leq 3\}$ as these are real items and $|\{s \in X_{23} \mid \Theta(s) = 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \leq b_2$. If t_3 is the first buyer, she gets $\{s \in X_{13} \mid \Theta(s) = 1\}$. Observe that the conditions of Claim 5 hold.

Subcase 5. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \leq b_2$ and $|\{s \in X_{23} \mid \Theta(s) = 1\}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| \leq b_3$.

To ensure 5(b) and 5(c). holds, we increase some prices the following way: If $|X_{23}| + |\{s \in X_{12} \mid \Theta(s) = 1\}| \geq b_2$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) > 1\} \cup X_{123}$ by ε. If $|X_{23}| + |\{s \in X_{12} \mid \Theta(s) = 1\}| < b_2$, we choose $b_2 - (|X_{23}| + |\{s \in X_{12} \mid \Theta(s) = 1\}|)$ items from $\{s \in X_{12} \mid \Theta(s) > 1\}$, and if the items in $\{s \in X_{12} \mid \Theta(s) > 1\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{12} \mid \Theta(s) > 1\} \cup X_{123}$ which were not chosen by ε. We do the same with b_3 instead of b_2: If $|X_{23}| + |\{s \in X_{13} \mid \Theta(s) = 1\}| \geq b_3$, we increase the prices in $\{s \in X_{13} \mid \Theta(s) > 1\} \cup X_{123}$ by ε. If $|X_{23}| + |\{s \in X_{13} \mid \Theta(s) = 1\}| > b_3$, we choose $b_3 - (|X_{23}| + |\{s \in X_{13} \mid \Theta(s) = 1\}|)$ items from $\{s \in X_{13} \mid \Theta(s) > 1\}$, and if the items in $\{s \in X_{13} \mid \Theta(s) > 1\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{13} \mid \Theta(s) > 1\} \cup X_{123}$ which were not chosen by ε. If we have to choose items from X_{123}, we start with the items which are already chosen, if there is any. If there is no chosen item or we have to choose more, we choose from the items with increased price, but we decrease their price by ε.

If t_1 comes first, she gets $\{s \in X_{12} \mid \Theta(s) \leq 3\}$ and $\{s \in X_{13} \mid \Theta(s) \leq 2\}$, as in $N_{G^+_{\pi+}}(t_1)$, we only increased the prices of items with label 4 or 5. It is easy to check for t_2 and t_3 that they take $\{s \in X_{12} \cup X_{23} \mid \Theta(s) \leq 2\}$ and $\{s \in X_{13} \cup X_{23} \mid \Theta(s) = 1\}$, respectively.

Case 5. The dummy items are in X_{123}.

By Lemma 4, a buyer has positive utility from her legal real items and negative utility from her non-legal items. We apply the same labeling procedure that we used in the proof of Theorem 8, that is, when the market satisfies property (OPT). Thus dummy items are now labeled by 5.

As before, t_i gets all items in $N_{G^+_{\pi^+}}(t_i)$ with label no greater than $4-i$, since these are real items. That implies 5(a) and 5(b). As $|N_{G^+_{\pi^+}}(t_i)| \geq b_i$ and t_i has positive utility for all real items in $N_{G^+_{\pi^+}}(t_i)$, 5(c) automatically holds. □

B. Bi-demand markets

For convenience, let S denote S^+. In the proof of Theorem 3, we showed the existence of an adequate ordering σ. In Lemma 3, we saw that, for $\delta := \Delta(\pi)/(|S|+1)$, setting the prices to $p(s) := \pi(s) + \delta \cdot \sigma(s)$ results in optimal dynamic pricing if (OPT) holds. However, when S contains dummy items beside the real ones, the pricing defined this way might not result in an optimal allocation. This is because when a buyer chooses items from her neighbors according to σ, the dummy items are not there in real life, therefore the buyer might skip dummy items in its neighborhood in $G^+_{\pi^+}$. As a consequence, she might take two items which are not allowed to her (that is, she takes two items from $N_{G^+_{\pi^+}}(Y)$ where $|N_{G^+_{\pi^+}}(Y)| \leq 2|Y|+1$ for some $\emptyset \neq Y \subsetneq T$) or she might take an item which is not feasible for her (that is, the item is not her neighbor in $G^+_{\pi^+}$). However, if we start with the minimum weighted covering π^+ described in Lemma 4, property 4(c) shows that if a buyer skips her dummy neighbors in the ordering, she does not take real items which are not legal for her as she has negative utility for them. That is, it is enough to ensure that if a buyer t has dummy neighbors, then she does not take two items from $N_{G^+_{\pi^+}}(Y)$ for every Y set with $|N_{G^+_{\pi^+}}(Y)| \leq 2|Y|+1$, $t \notin Y$ when she skips dummy items in the ordering. Recall that a set Y of buyers is dangerous if $|N_{G^+_{\pi^+}}(Y)| = 2|Y|+1$ and tight if $|N_{G^+_{\pi^+}}(Y)| = 2|Y|$. That means, we have to pay attention to dangerous and tight sets when pricing the items in the market.

The idea of the proof of Theorem 3 is the following: we set the prices to $p(s) := \pi(s) + \delta \cdot \sigma(s)$, where σ is an adequate ordering which is determined the same way as previously with the property (OPT). Then we increase some of the prices by ε to ensure that if a buyer has dummy items as neighbors, she will not take two items from $N_{G^+_{\pi^+}}(Y)$ if Y is a dangerous or tight set. We will use the following observations.

Observation 10.

(a) In $G^+_{\pi^+}$, the neighborhoods of dummy items are the same. As a result, for every $Y \subseteq T$, all dummy items are in $N_{G^+_{\pi^+}}(Y)$ or all of them are in $S - N_{G^+_{\pi^+}}(Y)$,

(b) A buyer $t \in \hat{T}$ has negative utility for a real item s if st is not legal, therefore buyers in $\hat{T}$ only take items that are feasible for them. Also, buyers in $T - \hat{T}$ take only feasible items as they have at least $b(t)$ neighbors in $G^+_{\pi^+}$,

(c) By the choice of ε, if we increase $p(s)$ by ε for some item s, the utility of $t \in \hat{T}$ for s becomes negative,

(d) By the choice of ε, if $t \in T - \hat{T}$, st is an edge in $G^+_{\pi^+}$, and we increase $p(s)$ by ε, the utility of t for s remains positive and still higher than for any s' where $s't$ is not legal.

Now we are ready to prove Theorem 3 without assuming (OPT).

Theorem 3. *Every bi-demand market admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

Proof. As before, we prove by induction on $|T|$, and the proof goes very similarly to the proof with the (OPT) assumption. The statement holds when $|T| = 1$, therefore $|T| \geq 2$ can be assumed.

Case 1. $|N_{G^+_{\pi+}}(Y)| \geq 2|Y| + 2$ for every $\emptyset \neq Y \subsetneq T$.

For any $t \in T$ and $s_1, s_2 \in N_{G^+_{\pi+}}(t)$, the graph $G_\pi - \{s_1, s_2, t\}$ still satisfies the conditions of Theorem 6(a), hence $\{s_1, s_2\}$ is feasible for t. Therefore σ can be chosen arbitrarily, since the current pricing ensures buyers will not buy items which are not optimal for them (Observation 10(b)).

Case 2. $|N_{G^+_{\pi+}}(Y)| \geq 2|Y| + 1$ for $\emptyset \neq Y \subsetneq T$ and there exists Y dangerous set, that is $|N_{G^+_{\pi+}}(Y)| = 2|Y| + 1$.

Let Z be an inclusionwise maximal dangerous set.

Subcase 2.1. There is no dangerous set disjoint from Z.

We have already shown in Section 5 that if a pair $s_1, s_2 \in N_{G^+_{\pi+}}(t)$ is not feasible for a buyer $t \in T - Z$, then $s_1, s_2 \in N_{G^+_{\pi+}}(Z)$.

First we consider the case when $|S - N_{G^+_{\pi+}}(Z)| \geq 2$. If $t \in T - Z$ and t has only one neighbor in $S - N_{G^+_{\pi+}}(Z)$, then for $T' = Z + t_0 \neq T$ we get $|N_{G^+_{\pi+}}(T')| = 2|T'|$. This case will be discussed later on (see Case 3). From now on, we assume that each $t \in T - Z$ has at least two neighbors in $N_{G^+_{\pi+}}(Z)$. Similarly as in the proof when (OPT) holds, let $t_0 \in T - Z$ be an arbitrary buyer who shares a neighbor with Z, and let $s_0 \in N_{G^+_{\pi+}}(t) \cap N_{G^+_{\pi+}}(Z)$. If it is possible, we choose t_0 and s_0 in such a way that s_0 is a dummy item. Let σ' be an arbitrary ordering of the items in $S - N_{G^+_{\pi+}}(Z)$, σ'' be an adequate ordering of the items in G'' where G'' is obtained by deleting the items in $S - (N_{G^+_{\pi+}}(Z) - s_0)$ and the buyers in $T - Z$, and σ''' be the trivial ordering of the single element set $\{s_0\}$. We consider the ordering $\sigma = (\sigma', \sigma'', \sigma''')$ of items in S.

Subcase 2.1.1. All dummy items are in $N_{G^+_{\pi+}}(Z)$.

If s_0 is dummy, any buyer from Z will choose items from $N_{G^+_{\pi+}}(Z)$ (see Observation 10(b)), but she will not take the dummy s_0. As σ'' was an adequate ordering of the items in G'', the remaining graph still admits a $(1, 2)$-factor. Buyers from $T - Z$ will take two real items from $S - N_{G^+_{\pi+}}(Z)$ as they have at least two neighbors in $S - N_{G^+_{\pi+}}(Z)$.

If s_0 is not dummy, we increase its price by ε. This way, buyers in Z who have dummy neighbors have negative utility for s_0, therefore such buyers will not take s_0 even after the deletion of the dummy items. If a buyer in Z has no dummy neighbors, she has at least two cheaper neighbors in $N_{G^+_{\pi+}}(Z)$ than s_0, which means that she will not take s_0 either. Again, a buyer from $T - Z$ will take two items from $S - N_{G^+_{\pi+}}(Z)$.

Subcase 2.1.2. All dummy items are in $S - N_{G^+_{\pi+}}(Z)$.

We increase all prices in $N_{G^+_{\pi+}}(Z)$ by ε. This way, if a buyer in $T - Z$ has less than two real neighbors in $S - N_{G^+_{\pi+}}(Z)$, she will not take items from $N_{G^+_{\pi+}}(Z)$ by Observation 10(c). For a buyer in Z, the order of neighbors in G_π remains unchanged and she still prefers items in $N_{G^+_{\pi+}}(Z)$ than items in $S - N_{G^+_{\pi+}}(Z)$ by Observation 10(d).

We finished the discussion of the case when there is no dangerous set disjoint from Z and $|S - N_{G^+_{\pi+}}(Z)| \geq 2$. Now let us assume that $|S - N_{G^+_{\pi+}}(Z)| = 1$, and let y_0 denote the single element in $S - N_{G^+_{\pi+}}(Z)$. As $|S| = 2|T|$, there is only one buyer in $T - Z$ (namely t_0). If t_0

has only two neighbors in $N_{G^+_{\pi+}}(Z)$, $X = \{t_0\}$ is a dangerous set disjoint from Z, contradicting the assumption of Subcase 2.1. Hence t_0 has at least three neighbors in $N_{G^+_{\pi+}}(Z)$. As before, s_0 denotes a neighbor of t_0 in $N_{G^+_{\pi+}}(Z)$. We define $\sigma = (\sigma', \sigma'', \sigma''')$ the same way as when $|S - N_{G^+_{\pi+}}(Z)| \geq 2$. First, we discuss the case when y_0 is a dummy item. Notice that y_0 is the only dummy item by Observation 10(a). Let y_1 denote the earliest neighbor of t_0 in $N_{G^+_{\pi+}}(Z)$ according to σ. Let $k \in \{1, \ldots, |S|\}$ denote the place of y_1 in the ordering. Then the price of y_1 is $\varepsilon + \delta \cdot k$ and t_0 has $\varepsilon - \delta \cdot k$ utility for y_1. We increase the price of every item in $N_{G^+_{\pi+}}(Z)$ by $\varepsilon - \delta \cdot \frac{2k+1}{2}$. As a result, t_0 has positive utility only for y_0 and y_1, while for buyers in Z, the utilities for their neighbors in $G^+_{\pi+}$ remain positive and the order of items remains unchanged. If t_0 is the first buyer, she takes y_0 and y_1, and any buyer in Z takes items according to σ. If y_0 is a real item, we do not change the prices. This way, t_0 takes at most one item from $N_{G^+_{\pi+}}(Z)$. A buyer from Z does not take y_0, since y_0 is feasible only for t_0, and she does not take s_0 which is at the end of the ordering.

Subcase 2.2. There exists a dangerous set disjoint from Z.

Let X be an inclusionwise minimal dangerous set disjoint from Z.

Subcase 2.2.1. For any $t \in X$ and for any $s_1, s_2 \in N_{G^+_{\pi+}}(t)$, the set $\{s_1, s_2\}$ is feasible.

We define an adequate ordering $\sigma := (\sigma', \sigma'')$ the same way as before with the (OPT) assumption. If the dummy items are in $N_{G^+_{\pi+}}(X)$, a buyer from $T - X$ who has no dummy neighbors chooses at most one item from $N_{G^+_{\pi+}}(X)$ (namely s_0), and a buyer from $T - X$ who has dummy neighbors also chooses at most one item from $N_{G^+_{\pi+}}(X)$, which is s_0 only if s_0 is real. If s_0 is dummy and the buyer has real neighbors in $N_{G^+_{\pi+}}(X)$, then she chooses one of them, but if she has only dummy neighbors in $N_{G^+_{\pi+}}(X)$, her utility is negative from the real items in $N_{G^+_{\pi+}}(X)$ by Observation 10(b), therefore she does not take anything from $N_{G^+_{\pi+}}(X)$. A buyer from X takes items from $N_{G^+_{\pi+}}(X)$, since if she has at least two real neighbors, she chooses two of them, but if she has at most one real neighbor, she does not take anything from $S - N_{G^+_{\pi+}}(X)$ as her utility is negative for them by Observation 10(b). If the dummy items are in $S - N_{G^+_{\pi+}}(X)$, we increase the prices in $N_{G^+_{\pi+}}(X) - \{s_0\}$ by ε. This way, a buyer from $T - X$ takes at most one item from $N_{G^+_{\pi+}}(X)$ (which is s_0), since if she has dummy neighbors, her utility is negative from $N_{G^+_{\pi+}}(X) - \{s_0\}$ by Observation 10(c), otherwise she has at least two cheaper real neighbors in $S - (N_{G^+_{\pi+}}(X) - \{s_0\})$. A buyer from X takes items from $N_{G^+_{\pi+}}(X)$ which does not cause a problem as X is an inclusionwise minimal dangerous set.

Subcase 2.2.2. There exists $t \in X$ and $s_1, s_2 \in N_{G^+_{\pi+}}(t)$ such that $\{s_1, s_2\}$ is not feasible.

In this case, we have already shown in the proof with assumption (OPT) that $X \cup Z = T$ and $N_{G^+_{\pi+}}(X) \cap N_{G^+_{\pi+}}(Z) = \{s_1, s_2\}$. We have defined an adequate ordering $\sigma := (\sigma', \sigma''|_{N_{G^+_{\pi+}}(X) - s_1}, \sigma''')$, where σ' and σ'' are adequate for the corresponding smaller graphs and σ''' is the trivial ordering of s_2. First, we assume s_1 and s_2 are both dummy. By Observation 10(a), that means there are no other dummy items. Any buyer in X takes items only from $N_{G^+_{\pi+}}(X) - \{s_1, s_2\}$, since if she has at least two real neighbors, she takes two of them, but if she has at most one real neighbor, she does not choose items which are not feasible for her (that is which are in $N_{G^+_{\pi+}}(Z) - \{s_1, s_2\}$) as her utility is negative for them by Observation 10(b). The reasoning is the same for buyers in Z. Now assume that s_1 is real and s_2 is dummy. We switch the roles of s_1 and s_2 if s_2

is real and s_1 is dummy. Now any buyer in X chooses items from $N_{G^+_{\pi^+}}(X) - \{s_2\}$ and any buyer in Z chooses items from $N_{G^+_{\pi^+}}(Z) - \{s_2\}$, since those buyers, who are in X and have dummy neighbor, have negative utility for $N_{G^+_{\pi^+}}(Z) - \{s_1, s_2\}$ and those buyers, who are in Z and have dummy neighbor, have negative utility for $N_{G^+_{\pi^+}}(X) - \{s_1, s_2\}$. Thirdly, if s_1 and s_2 are real items, we increase the price of s_2 by ε. This way, if all dummy items are in $N_{G^+_{\pi^+}}(Z)$, any buyer from Z takes at most one item from $N_{G^+_{\pi^+}}(Z) \cap N_{G^+_{\pi^+}}(X)$ (namely s_1), since buyers with dummy neighbors have negative utility for s_2 and buyers with only real neighbors have at least two cheaper neighbors in $N_{G^+_{\pi^+}}(X) - \{s_2\}$. Buyers in X do not take s_2 either as they have cheaper neighbors in $N_{G^+_{\pi^+}}(X) - \{s_2\}$. When all dummy items are in $N_{G^+_{\pi^+}}(X)$, the proof goes the same way.

Case 3. $|N_{G^+_{\pi^+}}(T')| = 2|T'|$ for some $\emptyset \neq T' \subsetneq T$.

As we showed in the case when (OPT) holds, there exists T' satisfying the assumption if and only if $G^+_{\pi^+}$ is not connected. Suppose $G^+_{\pi^+}$ has k components, and determine an adequate ordering for the components of $G^+_{\pi^+}$, separately. Then $\sigma = (\sigma', \sigma'', \dots \sigma^{(k)})$ is an adequate ordering for the whole graph, since if a buyer has no dummy neighbors, she has at least two real neighbors in her own component, and if a buyer has dummy neighbors, she has negative utility for all items which are not in her own component as Observation 10(b) shows. $\square$