Optimization over valuated matroids

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Modeling project work 3
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1 INTRODUCTION

Weighted matroid intersection problem is one of the major combinatorial optimization problems solvable in polynomial time. The problem generalizes a number of relatively unsophisticated problems such as the maximum-weight bipartite matching or minimum-weight arborescence problems. Consider the following general formulation known as the independent assignment problem proposed by Iri-Tomizawa.

Given a bipartite graph $G = (S_1, S_2, A)$, two matroids $M_1 = (S_1, \mathcal{B}_1)$ $M_2 = (S_2, \mathcal{B}_2)$ and a weight function $\omega : A \to R$, where (S_1, S_2) is the bipartition of the vertex set S of the graph G and A is the set of all the edges. The aim is to find a matching M from A which maximizes the following function:

$$\omega(M) = \sum \omega(a)|a \in M \tag{1.1}$$

that satisfies the following constraint:

$$d_1(M) \in \mathcal{B}_1 \qquad d_2(M) \in \mathcal{B}_2, \tag{1.2}$$

where $d_1(M)/d_2(M)$ represents the set of vertices in S_1/S_2 incident to M.

The independent assignment problem has been proved as a useful tool to formulate engineering problems in systems analysis.

As it has been mentioned in the previous project work, Dress-Wenzel introduced the notion of the valuation of the matroid. A **valuation** on a matroid $M = (S, \mathcal{B})$ is a function $\omega : \mathcal{B} \to R$ that satisfies the exchange property:

For any two bases B_1, B_2 and for any element $s_1 \in B_1 - B_2$, there exists an element $s_2 \in B_2 - B_1$ for which

$$\omega(B_1) + \omega(B_2) \leq \omega(B_1 - s_1 + s_2) + \omega(B_2 - s_2 + s_1) \tag{1.3}$$

A matroid equipped with such a function is called a **valuated matroid**.

A valuation ω can be derived from a weight function $\eta : S \to R$ and $\alpha \in R$ by the following relation:

$$\omega(B) = \alpha + \sum \eta(u)|u \in B, \qquad B \in \mathcal{B} \tag{1.4}$$

Such a valuation is called a separable valuation.

We consider an extension of the independent assignment problem to its valuated variant. More precisely, suppose that matroids $M_1 = (S_1, \mathcal{B}_1)$ and $M_2 = (S_2, \mathcal{B}_2)$ are given with valuations $\omega_1 : \mathcal{B}_1 \to R$ and $\omega_2 : \mathcal{B}_2 \to R$. Our goal is to find a matching $M \subseteq A$ that maximizes

$$\Omega(M) \equiv \omega(M) + \omega_1(d_1(M) + \omega_2((d_2(M))$$

subject to constraints (1.2).

This problem is called **the valuated independent assignment problem.**

In this report we formulate two variants of optimality criteria for valuated independent assignment problem by expanding them to the generally known optimality criteria to the independent valuated assignment problem.

The first type of the optimality criteria has been shown in terms of potentials (A. Frank, An algorithm for submodular functions on graphs, Annals of Discrete Mathematics,16 (1982), pp. 97–120)[3] and the second type- in terms of negative cycles in an auxiliary graph (S. Fujishige, A primal approach to the independent assignment problem, Journal of the Operations Research Society of Japan, 20 (1977), pp. 1–15.).[4]

It is worth to mention that a valuated matroid is an abstract representation of polynomial matrices (as it has been shown in the previous project work). On the other hand, it is known that polynomial matrices play a vital role in system engineering. Hence, it is natural to expect the effectiveness of valuated matroids in engineering problems.

In this regard, the considerable part of the applications of matroid theory is affiliated totally or partially to the matroid intersection problem.

2 Formulation of the problem. The problem can be interpreted as follows:

Assume that we are given a bipartite graph $G = (S_1, S_2 : A)$, valuated matroids $M_1 = (S_1, \mathcal{B}_1, \omega_1)$ and $M_2 = (S_2, \mathcal{B}_2, \omega_2)$, and a weight function $\omega : A \to R$. We consider the following optimization problem:

Find a matching $M(\subseteq A)$ that maximizes

$$\Omega(M) \equiv \omega(M) + \omega_1(d_1(M) + \omega_2((d_2(M))$$

subject to constraints (1.2).

It is obvious that we need two matroids of the same rank in order to be able to solve this problem. For more convenience we assume that the function ω, is a function on 2^S and finite on a basis $\mathcal{B}$ (and is $-\infty$ on every subset X of S which is not a basis, more precisely, $\omega_1(B) = \omega_2(B) = -\infty$ for $X \subseteq S$, where $B \notin \mathcal{B}_1$ or$B \notin \mathcal{B}_2$).

It is useful to mention the following optimization problems as well:

Now we suppose that we are given two valuated matroids $M_1 = (S, \mathcal{B}_1, \omega_1)$, $M_2 = (S, \mathcal{B}_2, \omega_2)$ defined on a common ground set S and a weight function $\omega : S \to R$.

Intersection problem. Find a common base $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ that maximizes

$$\omega(B) = \omega_1(B) + \omega_2(B)$$

Disjoint bases problem. Find disjoint bases B_1 and B_2 ($B_1 \cap B_2 = \emptyset$), where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, that maximize $\omega_1(B_1) + \omega_2(B_2)$

Partition problem. Find a partition $(B, S - B)$ of S such that it maximizes $\omega_1(B) + \omega_2(S - B)$.

The disjoint bases problem for more than two valuated matroids can also be interpreted as a valuated matroid independent assignment problem.

In addition to these variants of the independent assignment assignmemt problem, there is another optimization problem that can be formulated as follows:

For two matroids M_1 and M_2 defined on the same ground set S and two cost functions ω_1 and ω_2 defined on 2^S, we take into consideration the problem of finding bases B_1 and B_2 that minimize $\omega_1(B_1) + \omega_2(B_2)$ subject to some cardinality constraint on their intersection $B_1 \cap B_2$.

More specifically, suppose that we are given two matroids $M_1 = (S, B_1)$ and $M_2 = (S, B_2)$, ground set S, bases B_1 and B_2, two weight functions ω_1 and ω_2 and a nonnegative integer k.

Lendl, Peis, Timmermans have proposed the following types of weighted matroid intersection problem where the cardinality constraint is attached to the intersection of the bases [5].

a) Minimize $\omega_1(B_1) + \omega_2(B_2)$
subject to $B_i \in \mathcal{B}_{\rangle}$
$|B_1 \cap B_2| = k$

b) Minimize $\omega_1(B_1) + \omega_2(B_2)$
subject to $B_i \in \mathcal{B}_{\rangle}$
$|B_1 \cap B_2| \geq k$

c) Minimize $\omega_1(B_1) + \omega_2(B_2)$
subject to $B_i \in \mathcal{B}_{\rangle}$
$|B_1 \cap B_2| \leq k$

They showed that all of these 3 cases are strongly polynomial-time solvable problems: they introduced a new primal-dual algorithm for the case with equality constraint and reduced the other ones to a weighted matroid intersection problem.

Let us consider now some basic properties that were derived by Dress-Wenzel on the maximization of a valuated matroid.

Lemma 2.1 Let $B \in \mathcal{B}$. Then $\omega(B) \geq \omega(B^{'})$ for any $B^{'} \in \mathcal{B}$ if and only if

$$\omega(B, t, s) \leq 0 \text{ for any } (t, s) \text{ with } t \in C(B, s) \text{ [1]},$$

Remark: Suppose that $M = (S, \mathcal{B}, \omega)$ is a valuated matroid of the rank r. For $B \in \mathcal{B}$ and $s \in S - B$ the unique circuit contained in $B + s$ is denoted by $C(B, s)$. For $B \in \mathcal{B}$ and $s \in S - B$ and $t \in C(B, s)$ we have:

$$\omega(B, t, s) = \omega(B - t + s) - \omega(B).$$

The following lemma known as "Upper-bound lemma" gives a proof for Lemma 1.

Lemma 2.2 For $B, B^{'} \in \mathcal{B}$ the following inequality holds:

$$\omega(B^{'}) \leq \omega(B) + \omega(B, B^{'}) \text{ [1]}$$

Proof. For any $t_1 \in B - B^{'}$ there exists $s_1 \in B^{'} - B$ such that

$$\omega(B) + \omega(B^{'} \leq \omega(B - t_1 + s_1) + \omega(B^{'} + t_1 + s_1),$$

which, in turn, can be written as follows:

$$\omega(B^{'}) \leq \omega(B, t_1, s_1) + \omega(B_2^{'}),$$

where $B_2^{'} = B^{'} + t_1 - s_1$.

Then we get the following inequality:

$$\omega(B_2^{'}) \leq \omega(B, t_2, s_2) + \omega(B_3^{'}),$$

where $B_3^{'} = B_2^{'} + t_2 - s_2 = B^{'} - \{t_1, t_2\} + \{s_1, s_2\}$

So we get

$\omega(B^{'}) \leq \omega(B^{'}_3) + \sum_{i=1}^{2} \omega(B, t_i, s_i)$

By repeating this procedure, we can get the general inequality:

$\omega(B^{'}) \leq \omega(B) + \sum_{i=1}^{n} \omega(B, t_i, s_i) \leq \omega(B, B^{'})$, where $n = |B - B^{'}|$

2.1 Optimality criteria.

In this section we consider two optimality criteria for the valuated independent assignment problem on a graph $G = (S^{+}, S^{-}; A)$ with valuated matroids $M_1 = (S_1, \mathcal{B}_1, \omega_1)$, $M_2 = (S_2, \mathcal{B}_2, \omega_2)$ and a weight function $\omega : A \to R$

The first theorem uses the notion of the so-called potential function.

Theorem 2.1.1 An independent assignment M in G is optimal for the valuated independent assignment problem if and only if there exists a potential function $p : S_1 \cup S_2$ such that [1]

(1) $\omega(a) - p(d_1(a) + p(d_2(a))\begin{cases} \leq 0 & \text{if}(a \in A) \\ = 0 & (a \in M) \end{cases}$

(2) $d_1(M)$ is a maximum-weight base of M_1 with respect to $\omega_1[p_1]$,

(3) $d_2(M)$ is a maximum-weight base of M_2 with respect to $\omega_2[p_2]$,

where p_1/p_2 denotes the restriction of p to S_1/S_2 and $\omega_1[p_1]/\omega[p_2]$ denotes the similarity transformation, that is:

$$\omega_1[p_1](B_1) = \omega_1(B_1) + \sum p(u) | u \in B_1(B_1 \subseteq S_1),$$

$$\omega_2[p_2](B_2) = \omega_2(B_2) + \sum p(u) | u \in B_2(B_2 \subseteq S_2)$$

The following remark would be useful:

Remark.

For $p : S \to R$ we can define such a function $\omega[p] : \mathcal{B} \to R \cup \{-\infty\}$ that :

$$\omega[p](B) = \omega(B) + \sum \{p(u) | u \in B$$

This operation is called a **similarity transformation**.

Now suppose that p is a potential function that satisfies the abovementioned conditions (1)-(3) for some independent assignment M. Then this assignment is optimal if and only if it satisfies (1)-(3).

This optimality condition can also be reformulated in a form of Frank's weight splitting. Here we are given matroids $M^1 = (S, \mathcal{B}^1, \omega^1)$ $M^2 = (S, \mathcal{B}^2, \omega^2)$, and the goal is to maximize the following function:

$$\omega(B) + \omega^1(B) + \omega^2(B) \ (*)$$

Theorem 2.1.2 A common base B of $M^1 = (S, \mathcal{B}^1, \omega^1)$ and $M^2 = (S, \mathcal{B}^2, \omega^2)$ maximizes (*) if and only if there exist functions $\omega^1, \omega^2 : S \to R$ such that:

(1) $\omega(s) + \omega^1(s) + \omega^2(s)$ $(s \in S)$

(2)B is a maximum-weight base ofM_1 with respect to $\omega^1[\omega^1]$,

(3)B is a maximum-weight base ofM_2 with respect to $\omega^2[\omega^2]$ [1]

In order to describe the second optimality criterion we should introduce an auxilary graph $\tilde{G} = (\tilde{S}, \tilde{A})$ equipped with the same independent assignment M. We have $B_1 = d_1(M)$, $B_2 = d_2(M)$ and C_1, C_2 which denotes a fundamental circuit in M_1 and M_2, respectively.

The vertex set $\tilde{S}$ of a graph $\tilde{G}$ is defined as $\tilde{S} = S_1 \cup S_2$, while the the set of edges $\tilde{A}$ is $\tilde{A} = A_0 \cup M_0 \cup A_1 \cup A_2$, where each component is defined as follows:

$A_0 = \{a | a \in A\}$ -copy of A,

$M_0 = \{\overrightarrow{a} | a \in M\}$- $\overrightarrow{a}$ is the reorientation of a

$A_1 = \{(t, s) | t \in B_1, s \in S_1 - B_1\}$,

$A_2 = \{(t, s) | t \in B_2, s \in S_2 - B_2\}$.

Besides that, we need to introduce an arc length $\gamma(a)$, which is defined as:

$$\gamma(a) = \begin{cases} -\omega(a) & \text{if}(a \in A_0) \\ \omega(\overrightarrow{a} & (a = (t, s) \in M_0, \overrightarrow{a} = (s, t) \in M) \\ -\omega_1(B_1, t, s) & (a = (t, s) \in A_1) \\ -\omega_2(B_2, t, s) & (a = (s, t) \in A_2) \end{cases}$$

Remark. A directed cycle of negative length is called a negative cycle.

After givin the definitions above we can formulate the second optimality criterion.

Theorem 2.1.3 (Second optimality criterion). An independent assignment M of a graph G is optimal for the independent assignment problem if and only if there doesn't exist a negative cycle in an auxilary graph $\tilde{G}$ with respect to the arc length $\gamma(a)$. [1]

Remark. The second optimality criterion can be represented in the form of a Fenchel-type duality between the matroid valuations and their conjugate functions.

2.2 Matroid interection problem.

Let us recall the weighted matroid intersection problem.

We are given two matroids $M_1 = (S, B_1)$ and $M_2 = (S, B_2)$ defined on a common ground set S,bases B_1 and B_2, two weight functions ω_1 and ω_2 and a nonnegative integer k. We consider an optimization problem of finding a basis $X_1 \in \mathcal{B}_1$ and a basis $X_2 \in \mathcal{B}_2$ that minimize $\omega_1(X_1) + \omega_2(X_2)$ subject to a lower bound constraint $|X_1 \cap X_2| \geq k$, upper bound constraint $|X_1 \cap X_2| \leq k$, or the equality constraint $|X_1 \cap X_2| = k$ imposed on the intersection. First we consider the approach mentioned in [2].

It turns out that the problem with lower or upper bound constraint is computationally equal to matroid intersection, while the problem with equality constraint can be considered as a strictly more general problem.

If the equality constraint $|X_1 \cap X_2| = k$ is substituted either with the lower bound constraint $|X_1 \cap X_2| \geq k$ or with the upper bound constraint $|X_1 \cap X_2| \leq k$ the optimization problem is called $P_{\geq k}$ or $P_{\leq k}$, respectively.

It is worth to mention that we are interested only on those integers k, which have the range between 0 and $K = min\{r(M_1), r(M_2)\}$, where r is the rank of the matroid M (cardinality of each basis in M).

As it was mentioned before Lendl et al. proposed the following solution:

They polynomially reduced both $P_{\geq k}$ or $P_{\leq k}$ to a weighted matroid intersection, which can be solved in strongly-polynomial time. This, in turn, leads us to the question if the problem with equality constraint can be solved in strongly-polynomial time.

Theorem 2.2.1 Both $P_{\geq k}$ and $P_{\leq k}$ can be reduced to a weighted matroid intersection.[2]

Let us describe the algorithm proposed by Lend et al:

First we need to solve the following problem without any constraint imposed on the intersection:

$$min\{\omega_1(X_1) + \omega_2(X_2) | X_1 \in \mathcal{B}_1, X_2 \in \mathcal{B}_2\}$$

Suppose that $(\overset{*}{X}_1, \overset{*}{X}_2)$ is an optimal solution of the problem. Then:

1) If $|X_1 \cap X_2| = k$, then we are done since $(\overset{*}{X}_1, \overset{*}{X}_2)$ is optimal solution for $P_{=k}$.

2) Else, if $|X_1 \cap X_2| = k' < k$, then the algorithm takes $(\overset{*}{X}_1, \overset{*}{X}_2)$ for $P_{=k'}$ as a starting point and increases k' by one until it reaches k.

3) Else, if $|X_1 \cap X_2| > k$, then we consider the case $P_{=\overset{*}{k}}$ where $\overset{*}{k} = r(M_1) - k$, costs ω_1 and $\omega_2 = -\overset{*}{\omega}_2$ and the matroids $M_1 = (S, \mathcal{B}_1)$ $M_2 = (S, \overset{*}{\mathcal{B}}_2)$. Then an optimal solution $(X_1, S \setminus X_2)$ of problem $P_{=\overset{*}{k}}$ is compatible with the solution (X_1, X_2) of the original problem.

Theorem 2.2.2(Optimality condition)

For fixed $\lambda \geq 0$, the pair $(X_1, X_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ is a minimizer of $val(\lambda)$ if there exist $\alpha, \beta \in R_+^{|S|}$ such that:

(a) X_1 is a min cost basis for $\omega_1 - \alpha$ and X_2 is the min cost basis for $\omega_2 - \beta$,

(b) $\alpha_s = 0$ [2]

In order to understand the essence of λ, let us consider the following linear concave function:

$$val(\lambda) = min(\omega_1(X_1) + \omega_2(X_2)\text{-}\lambda|X_1 \cap X_2|)$$

which is dependent on λ-parameter. Here $val(\lambda) + k\lambda$ is the Lagrangian relaxation of the original problem with the equality constraint on the intersection.

The next question is how can we construct an auxilary graph $\tilde{G}$?

Let us consider a sequence $(X_1, X_2, \alpha, \beta, \lambda)$ that satisfies the optimality conditions in Theorem 2.2.2. Then we can construct an auxilary digraph $\tilde{G} = \tilde{G}((X_1, X_2, \alpha, \beta, \lambda)$ with red-blue colored edges as follows:

1) 1 vertex stands for each element in S;

2) red edge (s, t) if $s \notin X_1, X_1 - t + s \in \mathcal{B}_1$, $\omega_1(s) - \alpha_s = \omega_1(t) - \alpha_t$;

3) blue edge (t, v) if $s \notin X_2, X_2 - t + v \in \mathcal{B}_2$, $\omega_1(v) - \beta_v = \omega_2(t) - \beta_t$

Here we should note that any red edge (s, t) stands for a move in $\mathcal{B}_1$ from X_1 to $X_1 \cup \{s\} \setminus \{t\} \in \mathcal{B}_1$.
Similarly, any blue edge (s, t) stands for a move in $\mathcal{B}_2$ from X_2 to $X_2 \cup \{t\} \setminus \{s\} \in \mathcal{B}_2$.

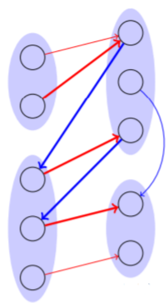

Figure 1

The following lemma would be useful for defining an augmenting path in this digraph:
Lemma 2.2.1 Suppose that we are given a matroid $M = (S, \mathcal{B}$ with weight function $\omega : S \to R$ and $X \in \mathcal{B}$. Let $x_1, x_2, ..., x_m \in X$ and $y_1, y_2, ..., y_m \notin X$ satisfying the following conditions:
a) $X + y_j - x \in \mathcal{B}$ and $\omega(x_j) = \omega(y_j)$ for $j = 1, ..., m$
b) $X + y_j - x_i \notin \mathcal{B}$ and $\omega(x_i)\,\omega(y_j)$ for $i \leq j, j \leq 1, i \neq j$

Then $X \setminus \{x_1, x_2, ..., x_m\} \cup \{y_1, y_2, ..., y_m\} \in \mathcal{B}$ [8]

Definition 2.2.1. Any shortest red-blue alternating path connecting the vertex in $X_2 \setminus X_1$ with the vertex in $X_1 \setminus X_2$ is called an augmenting path.

Lemma 2.2.2 Let us denote an augmenting in $\tilde{G}$ by P, then:
1) $X_1{}' = X_1 \bigoplus P$ is a min cost basis with respect to $\omega_1 - \alpha$,
2)$X_2{}' = X_2 \bigoplus P$ is a min cost basis with respect to $\omega_2 - \beta$
3) $|X_1{}^{'} \cap X_2{}^{'}| = |X_1 \cap X_2| + 1$

Primal update.

Suppose that we are given a tuple $(X_1, X_2, \alpha, \beta, \lambda)$ that satisfies the optimality conditions stated above. Then we update (X_1, X_2) to $(X_1{}^{'}, X_2{}^{'})$ by $X_1{}' = X_1 \bigoplus P$ and $X_2{}' = X_2 \bigoplus P$ as long as an augmenting path allows us (i.e., there exists an augnmenting path in the graph).

3 Algorithmic solution of the problem.

Dual update.

If there is no augmenting path in the graph and $|X_1 \cap X_2| < k$, we denote by R the set of vertices that are reachable from $X_2 \setminus X_1$ on some red-blue alternating path. Here we should take into account that $X_2 \setminus X_1 \subseteq R$ and $(X_1 \setminus X_2) \cap R = \emptyset$. Then we define residual cost functions for each $s \in S$ as follows:

$$\overline{\omega_1}(s) = \omega_1(s) - \alpha_s \text{ and } \overline{\omega_2}(s) = \omega_2(s) - \beta_s$$

It is worth to mention that by optimality of X_1 and X_2 with respect to $\overline{\omega_1}$ and $\overline{\omega_2}$ we have $\overline{\omega_1}(s) \geq \overline{\omega_1}(t)$ whenever $X_1 - t + s \in \mathcal{B}_1$, and $\overline{\omega_2}(s) \geq \overline{\omega_2}(t)$ whenever $X_2 - t + s \in \mathcal{B}_2$.

We define a new notion of "step length" $\delta > 0$, which is computed as follows:

$$\delta_1 = min\{\overline{\omega_1}(s) - \overline{\omega_1}(t) | s \in R \setminus X_1, t \in X_1 \setminus R, X_1 - t + s \in \mathcal{B}_1\}$$
$$\delta_2 = min\{\overline{\omega_2}(v) - \overline{\omega_2}(t) | v \notin X_2 \cup R, t \in X_2 \cap R, X_1 - v + t \in \mathcal{B}_2\}$$

In some cases it is possible that sets over which minimum is calculated are empty sets. In this case we suppose that this minimum is equal to ∞. In the particular case when $M_1 = M_2$ this situation can not occur.

Since neither a red, nor a blue arc goes from R to $S \setminus R$, we know that bot δ_1 and δ_2 are strictly positive, i.e., $\delta = min\{\delta_1, \delta_2\} > 0$. Now we have:

$$\alpha_s^{'} = \begin{cases} \alpha_s + \delta & \text{if}(s \in R) \\ \alpha_s & \text{otherwise} \end{cases} \; and$$

$$\beta_s^{'} = \begin{cases} \beta_s & \text{if}(s \in R) \\ \beta_s + \delta & \text{otherwise} \end{cases}$$

Lemma 3.1 $(X_1, X_2, \alpha^{'}, \beta^{'})$ satisfies the optimality condition when $\lambda^{'} = \lambda + \delta$

Lemma 3.2 If $(X_1, X_2, \alpha, \beta, \lambda)$ satisfies the optimality conditions and the "step length" $\delta < \infty$, then primal update can be performed after at most $|S|$ iterations of dual update.

Summing up the abovementioned facts, we get the following algorithm named **primal-dual algorithm** proposed by Lendl et al.

Input: $M_1 = (S, \mathcal{B}_1)$, $M_2 = (S, \mathcal{B}_2)$, $\omega_1, \omega_2 : S \to R, k \in N$
Output: Optimal solution (X_1, X_2) of the problem $P_{=k}$
Step1. Calculate the optimal solution (X_1, X_2) of

$$min\{\omega_1(X_1) + \omega_2(X_2) | X_1 \in \mathcal{B}_1, X_2 \in \mathcal{B}_2\}.$$

Step2. If $|X_1 \cap X_2| = k$, return (X_1, X_2) as an optimal solution.
Step3. Else $|X_1 \cap X_2| > k$, run algorithm on $M_1, \overset{*}{M_2}$, ω_1 and $\omega_2 = -\overset{*}{\omega}_2$, $\overset{*}{k} = r(M_1) - k$.
Step4. Else define $\lambda := 0, \alpha := 0, \beta := 0$.
Step5. While $|X_1 \cap X_2| < k$, do:

1)Construct an auxilary graph based on a tuple $(X_1, X_2, \alpha, \beta, \lambda)$
2) If there is an augmenting path in this graph, then update primal :

$$X_1{}' = X_1 \bigoplus P,\ X_2{}' = X_2 \bigoplus P$$

3) Else compute step length δ as it was calculated above.
If $\delta = \infty$, then the algorithms returns an infeasible solution.
Else set $\lambda := \lambda + \delta$ and update dual:

$$\alpha_s^{'} = \begin{cases} \alpha_s + \delta & \text{if the vertex } s \text{ is reachable} \\ \alpha_s & \text{otherwise} \end{cases} \quad and$$

$$\beta_s^{'} = \begin{cases} \beta_s & \text{if the vertex } s \text{ is reachable} \\ \beta_s + \delta & \text{otherwise} \end{cases}$$

Iterate the process with $(X_1, X_2, \lambda, \alpha, \beta)$

Step6. Return (X_1, X_2).

It is worth to mention the following statement:
The primal-dual algorithm solves $(P_{=k})$ using at most $k \times |S|$ primal or dual augmentations. Furthermore, the sequence of optimal solutions (X_k, Y_k) for all $(P_{=k})$ with $k = 0, 1, ..., K$ can be computed for $|S|^2$ primal or dual augmentations.

There is another solution, which is much simpler in comparison to the algorithm developed by S. Lendl, B. Peis, V. Timmermans. It was proposed by Laszlo Vegh:
Consider first the following auxiliary problem. For matroids M_1 and M_2 on S, we asssume that they admit bases B_1 and B_2 for which $|B_1 \cup B_2| \subseteq k$. Find a basis B_1 of M_1 and a basis B_2 of M_2 for which $|B_1 \cup B_2| \leq k$ and $\tilde{\omega}_1(B_1) + \tilde{\omega}_2(B_2)$ is as small as possible.

In order to solve this problem, let S_1 and S_2 be two disjoint copies of S. Put M_1 equipped with ω_1 on S_1 and put M_2 equipped with ω_2 on S_2. Then suppose that M is the direct sum of these two matroids. Let $M^{'}$ be a matroid on $S_1 \cup S_2$ in which a subset is independent if it includes at most k pairs where a pair consists of the copies $s_1 \in S_1$ and $s_2 \in S_2$ associated with an element s of S. This is indeed a matroid and the auxiliary problem is nothing else but finding a cheapest common independent set of r_1+r_2 elements in these two matroids M and $M^{'}$. Returning to the original problem, with the help of the auxiliary problem, we find bases B_1 and B_2 for which $|B_1 \cap B_2| \leq k$ and $\tilde{\omega}_1(B_1)+\tilde{\omega}_2(B_2)$ is minimum. If $|B_1 \cap B_2| = k$, then we are done. So we may assume that (1) $|B_1 \cap B_2| < k$. We claim that B_1 is a minimum ω_1-cost basis of M_1. Indeed, if this is not the case, then, by the standard characterization of cheapest bases, there are elements $t \in B_1$, $s \in S\text{-}B_1$ for which $\omega_1(s) < \omega_1(t)$ and $B_{1a} := B_1 - t + s$ is a basis of M_1. But in this case (1) implies that $|B_1 \cap B_2| \leq k$, contradicting the minimality of $\tilde{\omega}_1(B_1) + \tilde{\omega}_2(B_2)$. Similarly, we see that B_2 is a minimum ω_2-cost basis of M_2. Symmetrically, we can find bases $B_1^{'}$ and $B_2^{'}$ for which $|B_1^{'} \cap B_2^{'}| \geq k$ and $\omega_1(B_1^{'}) + \omega_2(B_2^{'})$ minimum. If $|B_1^{'} \cap B_2^{'}| = k$, we are done.

So we may assume that (2)

$|B_1^{'} \cap B_2^{'}| > k$. In this case it is also true that $B_1^{'}$ is a minimum ω_1-cost basis of M_1 and that $B_2^{'}$ is a minimum ω_2-cost basis of M_2. It is known that the minimum cost bases of a matroid form a matroid. This implies that, starting from B_1, we can arrive at $B_1^{'}$ by a sequence of element-changes so that each intermediate basis of M_1 is a minimum ω_1-cost basis of M_1. After this, starting from B_2, we can arrive at $B_2^{'}$ by a sequence of element-changes so that each intermediate

basis of M_2 is a minimum ω_2-cost basis of M_2. But then there must be an intermediate pair of bases for which the intersection of the two bases has exactly k elements.

As it was mentioned above, this solutuion is much more simple than LPT algorithm. The algorithm of Lendl et al. requires at most $|S|$ updates of potential functions, therefore one update of a solution takes $O(|S|^2 r)$ time, since each update of potential functions requires $O(|S|r)$ time.

Bibliography

[1] KAZUO MUROTA VALUATED MATROID INTERSECTION, I: OPTIMALITY CRITERIA, SIAM J. Discrete Mathematics, Vol.9 (1996), 545-561

[2] Stefan Lendl · Britta Peis · Veerle Timmermans Matroid bases with cardinality constraints on the intersection Mathematical Programming, 2021

[3] A. Frank, An algorithm for submodular functions on graphs, Annals of Discrete Mathematics,16 (1982), pp. 97–120

[4] S. Fujishige, A primal approach to the independent assignment problem, Journal of the Operations Research Society of Japan, 20 (1977), pp. 1–15.

[5] Y. Iwamasa K. Takazawa Optimal matroid bases with intersection constraints: Valuated matroids, M-convex functions, and their applications, Mathematical Programming, 1-28 (2021)

[6] A. W. M. Dress and W. Wenzel. Valuated matroids: A new look at the greedy algorithm. Applied Mathematics Letters, 3(2):33–35, 1990

[7] A. W. M. Dress and W. Wenzel. Valuated matroids. Advances in Mathematics, 93:214–250, 1992.

[8] A. Frank. A weighted matroid intersection algorithm. Journal of Algorithms, 2:328–336, 1981