Optimization over valuated matroids

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Content

Introduction

Weighted matroid intersection problem is one of the major combinatorial optimization problems solvable in polynomial time. The independent assignment problem (Iri-Tomizawa):
Given a bipartite graph $G = (S_1, S_2, A)$, two matroids $M_1 = (S_1, \mathcal{B}_1)$ $M_2 = (S_2, \mathcal{B}_2)$ and a weight function $\omega : A \to R$, where (S_1, S_2) is the bipartition of the vertex set S of the graph G and A is the set of all the edges. The aim is to find a matching M from A which maximizes the following function:

$$\omega(M) = \sum \omega(a) | a \in M \tag{1.1}$$

and satisfies the following:

$$d_1(M) \in \mathcal{B}_1 \qquad d_2(M) \in \mathcal{B}_2, \tag{1.2}$$

where $d_1(M)/d_2(M)$ represents the set of vertices in S_1/S_2 incident to M. A valuation on a matroid $M = (S, \mathcal{B})$ is a function $\omega : \mathcal{B} \to R$ that satisfies the exchange property:

For any two bases B_1, B_2 and for any element $s_1 \in B_1 - B_2$, there exists an element $s_2 \in B_2 - B_1$ for which

$$\omega(B_1) + \omega(B_2) \leq \omega(B_1 - s_1 + s_2) + \omega(B_2 - s_2 + s_1) \tag{1.3}$$

A matroid equipped with such a function is called a valuated matroid.

Suppose that matroids $M_1 = (S_1, \mathcal{B}_1)$ and $M_2 = (S_2, \mathcal{B}_2)$ are given with valuations $\omega_1 : \mathcal{B}_1 \to R$ and $\omega_2 : \mathcal{B}_2 \to R$. Our goal is to find a matching $M \subseteq A$ that maximizes

$$\Omega(M) \equiv \omega(M) + \omega_1(d_1(M) + \omega_2((d_2(M))$$

subject to constraints (1.2).
This problem is called **the valuated independent assignment problem.**

Formulation of the problem

Assume that we are given a bipartite graph $G = (S_1, S_2 : A)$, valuated matroids $M_1 = (S_1, \mathcal{B}_1, \omega_1)$ and $M_2 = (S_2, \mathcal{B}_2, \omega_2)$, and a weight function $\omega : A \to R$. We consider the following optimization problem:
Find a matching $M(\subseteq A)$ that maximizes

$$\Omega(M) \equiv \omega(M) + \omega_1(d_1(M) + \omega_2((d_2(M))$$

subject to constraints (1.2).

Lendl, Peis, Timmermans have proposed the following types of weighted matroid intersection problem where the cardinality constraint is attached to the intersection of the bases.

a) Minimize $\omega_1(B_1) + \omega_2(B_2)$
subject to $B_i \in \mathcal{B}_i$
$|B_1 \cap B_2| = k$

b) Minimize $\omega_1(B_1) + \omega_2(B_2)$
subject to $B_i \in \mathcal{B}_i$
$|B_1 \cap B_2| \geq k$

c) Minimize $\omega_1(B_1) + \omega_2(B_2)$
subject to $B_i \in \mathcal{B}_i$
$|B_1 \cap B_2| \leq k$

Optimality criteria

We consider two optimality criteria for the valuated independent assignment problem on a graph $G = (S^+, S^-; A)$ with valuated matroids $M_1 = (S_1, \mathcal{B}_1, \omega_1)$, $M_2 = (S_2, \mathcal{B}_2, \omega_2)$ and a weight function $\omega : A \to R$

Theorem 1 An independent assignment M in G is optimal for the valuated independent assignment problem if and only if there exists a potential function $p : S_1 \cup S_2$ such that

(1) $\omega(a) - p(d_1(a) + p(d_2(a)) \begin{cases} \leq 0 & \text{if}(a \in A) \\ = 0 & (a \in M) \end{cases}$

(2) $d_1(M)$ is a maximum-weight base of M_1 with respect to $\omega_1[p_1]$,

(3) $d_2(M)$ is a maximum-weight base of M_2 with respect to $\omega_2[p_2]$

In order to describe the second optimality criterion we should introduce an auxilary graph $\tilde{G} = (\tilde{S}, \tilde{A})$ equipped with the same independent assignment M. The vertex set $\tilde{S}$ of a graph $\tilde{G}$ is defined as $\tilde{S} = S_1 \cup S_2$, while the the set of edges $\tilde{A}$ is $\tilde{A} = A_0 \cup M_0 \cup A_1 \cup A_2$, where each component is defined as follows:

$A_0 = \{a | a \in A\}$ -copy of A,

$M_0 = \{\overrightarrow{a} | a \in M\}$- $\overrightarrow{a}$ is the reorientation of a

$A_1 = \{(t, s) | t \in B_1, s \in S_1 - B_1\}$,

$A_2 = \{(t, s) | t \in B_2, s \in S_2 - B_2\}$.

Besides that, we need to introduce an arc length $\gamma(a)$, which is defined as:

$$\gamma(a) = \begin{cases} -\omega(a) & \text{if}(a \in A_0) \\ \omega(\overrightarrow{a} & (a = (t, s) \in M_0, \overrightarrow{a} = (s, t) \in M) \\ -\omega_1(B_1, t, s) & (a = (t, s) \in A_1) \\ -\omega_2(B_2, t, s) & (a = (s, t) \in A_2) \end{cases}$$

Remark. A directed cycle of negative length is called a negative cycle.

Theorem 2 (Second optimality criterion). An independent assignment M of a graph G is optimal for the independent assignment problem if and only if there doesn't exist a negative cycle in an auxilary graph $\tilde{G}$ with respect to the arc length $\gamma(a)$.

Matroid intersection problem

Two matroids $M_1 = (S, B_1)$ and $M_2 = (S, B_2)$ defined on a common ground set S,bases B_1 and B_2, two weight functions ω_1 and ω_2 and a nonnegative integer k. We consider an optimization problem of finding a basis $X_1 \in \mathcal{B}_1$ and a basis $X_2 \in \mathcal{B}_2$ that minimize $\omega_1(X_1) + \omega_2(X_2)$ subject to a lower bound constraint $|X_1 \cap X_2| \geq k$, upper bound constraint $|X_1 \cap X_2| \leq k$, or the equality constraint $|X_1 \cap X_2| = k$ imposed on the intersection.

The problem with lower or upper bound constraint is computationally equal to matroid intersection, while the problem with equality constraint can be considered as a strictly more general problem.
If the equality constraint $|X_1 \cap X_2| = k$ is substituted either with the lower bound constraint $|X_1 \cap X_2| \geq k$ or with the upper bound constraint $|X_1 \cap X_2| \leq k$ the optimization problem is called $P_{\geq k}$ or $P_{\leq k}$, respectively.
It is worth to mention that we are interested only on those integers k, which have the range between 0 and $K = min\{r(M_1), r(M_2)\}$, where r is the rank of the matroid M (cardinality of each basis in M).

Lendl et al. proposed the following solution:
They polynomially reduced both $P_{\geq k}$ or $P_{\leq k}$ to a weighted matroid intersection, which can be solved in strongly-polynomial time. This, in turn, leads us to the question if the problem with equality constraint can be solved in strongly-polynomial time.

Theorem Both $P_{\geq k}$ and $P_{\leq k}$ can be reduced to a weighted matroid intersection.

First we need to solve the following problem without any constraint imposed on the intersection:

$$min\{\omega_1(X_1) + \omega_2(X_2) | X_1 \in \mathcal{B}_1, X_2 \in \mathcal{B}_2\}$$

Suppose that $(\overset{*}{X}_1, \overset{*}{X}_2)$ is an optimal solution of the problem. Then:

1) If $|X_1 \cap X_2| = k$, then we are done since $(\overset{*}{X}_1, \overset{*}{X}_2)$ is optimal solution for $P_{=k}$.

2) Else, if $|X_1 \cap X_2| = k' < k$, then the algorithm takes $(\overset{*}{X}_1, \overset{*}{X}_2)$ for $P_{=}k'$ as a starting point and increases k' by one until it reaches k.

3) Else, if $|X_1 \cap X_2| > k$, then we consider the case $P_{=\overset{*}{k}}$ where $\overset{*}{k} = r(M_1) - k$, costs ω_1 and $\omega_2 = -\overset{*}{\omega}_2$ and the matroids $M_1 = (S, \mathcal{B}_1)$ $M_2 = (S, \overset{*}{\mathcal{B}}_2)$. Then an optimal solution $(X_1, S \setminus X_2)$ of problem $P_{=\overset{*}{k}}$ is compatible with the solution (X_1, X_2) of the original problem.

Optimality condition

For fixed $\lambda \geq 0$, the pair $(X_1, X_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ is a minimizer of $val(\lambda)$ if there exist $\alpha, \beta \in R_+^{|S|}$ such that:

(a) X_1 is a min cost basis for $\omega_1 - \alpha$ and X_2 is the min cost basis for $\omega_2 - \beta$,

(b) $\alpha_s = 0$

$$val(\lambda) = min(\omega_1(X_1) + \omega_2(X_2)\text{-}\lambda|X_1 \cap X_2|)$$

LPT algorithm

The following algorithm named **primal-dual algorithm** proposed by Lendl et al.

Input: $M_1 = (S, \mathcal{B}_1)$, $M_2 = (S, \mathcal{B}_2)$, $\omega_1, \omega_2 : S \to R, k \in N$
Output: Optimal solution (X_1, X_2) of the problem $P_{=k}$
Step1. Calculate the optimal solution (X_1, X_2) of

$$min\{\omega_1(X_1) + \omega_2(X_2) | X_1 \in \mathcal{B}_1, X_2 \in \mathcal{B}_2\}.$$

Step2. If $|X_1 \cap X_2| = k$, return (X_1, X_2) as an optimal solution.

Step3. Else $|X_1 \cap X_2| > k$, run algorithm on $M_1, \overset{*}{M_2}$, ω_1 and $\omega_2 = -\overset{*}{\omega}_2$, $\overset{*}{k} = r(M_1) - k$.
Step4. Else define $\lambda := 0, \alpha := 0, \beta := 0$.

Step5. While $|X_1 \cap X_2| < k$, do:

1)Construct an auxilary graph based on a tuple $(X_1, X_2, \alpha, \beta, \lambda)$
2) If there is an augmenting path in this graph, then update primal :

$$X_1' = X_1 \bigoplus P,\ X_2' = X_2 \bigoplus P$$

Iterate the process with $(X_1, X_2, \lambda, \alpha, \beta)$

Step6. Return (X_1, X_2).

The primal-dual algorithm solves $(P_{=k})$ using at most $k \times |S|$ primal or dual augmentations. Furthermore, the sequence of optimal solutions (X_k, Y_k) for all $(P_{=k})$ with $k = 0, 1, ..., K$ can be computed for $|S|^2$ primal or dual augmentations.

Végh's approach

There is another solution, which is much simpler in comparison to the algorithm developed by S. Lendl, B. Peis, V. Timmermans. It was proposed by Laszlo Végh:

Consider first the following auxiliary problem. For matroids M_1 and M_2 on S, we assume that they admit bases B_1 and B_2 for which $|B_1 \cup B_2| \subseteq k$. Find a basis B_1 of M_1 and a basis B_2 of M_2 for which $|B_1 \cup B_2| \leq k$ and $\tilde{\omega}_1(B_1) + \tilde{\omega}_2(B_2)$ is as small as possible.

Let S_1 and S_2 be two disjoint copies of S. Put M_1 equipped with ω_1 on S_1 and put M_2 equipped with ω_2 on S_2. Then suppose that M is the direct sum of these two matroids. Let $M^{'}$ be a matroid on $S_1 \cup S_2$ in which a subset is independent if it includes at most k pairs where a pair consists of the copies $s_1 \in S_1$ and $s_2 \in S_2$ associated with an element s of S. This is indeed a matroid and the auxiliary problem is nothing else but finding a cheapest common independent set of $r_1 + r_2$ elements in these two matroids M and $M^{'}$.

This approach is much more simple than LPT algorithm. The algorithm of Lendl et al. requires at most $|S|$ updates of potential functions, therefore one update of a solution takes $O(|S|^2 r)$ time, since each update of potential functions requires $O(|S| r)$ time.

Thank you for your attention!