Neural networks for boundary value problems

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Outline

- Methods of fundamental solution(MFS) with two approaches by neural network
- Convergence analysis of the MFS
- Numerical results
- Future works

Problem

$$\begin{cases} Lu = 0 \quad \textit{on } \Omega \\ u|_{\Gamma_1} = g \quad \textit{and} \quad \frac{\partial u}{\partial n}\Big|_{\Gamma_2} = f. \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ *is an open set and* $\partial\Omega$ *consists of two disjoint parts,* Γ_1 *and* Γ_2*, such that* $\partial\Omega = \Gamma_1 \cup \Gamma_2$*.*

Definition

A fundamental solution to a linear differential operator L is a distribution E such that $L(E) = \delta$

Fundamental solution of Laplace operator

$$\phi_y(x) = \begin{cases} -\frac{1}{2\pi}\log(|x-y|) & \text{for } d = 2 \\ \frac{1}{(d-2)\omega_n \|x-y\|^{d-2}} & \text{for } d \geq 3 \end{cases}.$$

Method of fundamental solution

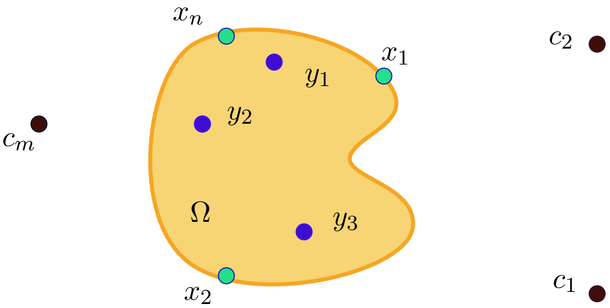


Figure: Inner, Boundary, Outer points

Method of fundamental solution

According to the single layer representation,

$$u(x) = \int_{\partial\Omega} \phi_y(x)\mathcal{G}(y)\,\mathrm{d}y. \tag{2}$$

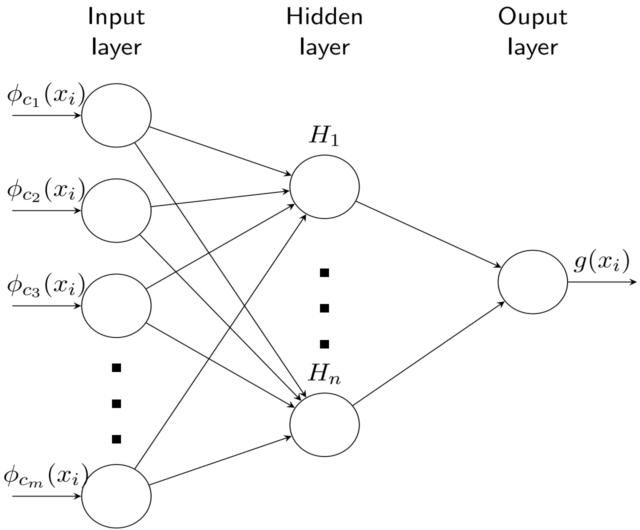
If x are boundary point, $\phi_y(x)$ was not well-defined (singular integrals).
Hence auxiliary boundary Ω' was introduced

$$u(x) = \int_{\partial\Omega'} \phi_y(x)\mathcal{G}'(y)\,\mathrm{d}y$$

Discretizing the integrals, we get

$$u(x) \approx \sum_{j=1}^{m} a_j \phi_{c_j}(x) \tag{3}$$

Method of fundamental solution



The first approach

- Dirichlet typed input:
  $(\phi_{c_1}(x_i), \cdots, \phi_{c_m}(x_i))$
  where $x_i \in \Gamma_1$
  Neumann typed input :
  $\left(\frac{\partial}{\partial \mathbf{n}}\phi_{c_1}(x_i), \cdots, \frac{\partial}{\partial \mathbf{n}}\phi_{c_m}(x_i)\right)$
  where $x_i \in \Gamma_2$
- Dirichlet typed output :
  $g(x_i)$ where $x_i \in \Gamma_1$
  Neumann typed output:
  $f_{\mathbf{n}}(x_i)$ where $x_i \in \Gamma_2$

The second approach

- Input

$$(\phi_{c_j}(x_1), \cdots, \phi_{c_j}(x_n), \frac{\partial}{\partial \mathbf{n}}\phi_{c_j}(x_1'), \cdots, \frac{\partial}{\partial \mathbf{n}}\phi_{c_j}(x_n'))$$

- Output $\phi_{c_j}(y)$

Method of fundamental solution

The first approach

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The second approach

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- Output $\phi_{c_j}(y)$

(*Katsurada and Okamoto ('96), and Fairweather and Karageorghis ('98)*) For the Dirichlet problem for the Laplace equation on the circle $(0, \rho)$, boundary function is analytical, solution is analytically harmonic continuable to the whole plane:

$$\|u - u_M\|_{L^\infty(\Omega)} \leq C \left(\frac{\rho}{R}\right)^M$$

(*Kitagawa ('88, '91)*)u is not analytically continuable to the whole plane, but rather only up to an extension $B(0; r_0)$

$$\|u - u_M\|_{L^\infty(\Omega)} \leq$$

$$\|u\|_{L^\infty(\partial B(0,r_0))} \left(\frac{2}{1 - \frac{\rho}{R}}\right) \left[(1 + A(R, p)) \left(\frac{\rho}{r_0}\right)^{M/3} + 4 \left(\frac{\rho}{R}\right)^{M/3}\right]$$

where $A(R, \rho)$ is some constant between 1 and 2.

Convergence analysis of the MFS

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Numerical results

- What is the optimal **distribution** of the boundary and outer points?
- What is the optimal **distance** of the outer points from the boundary?
- What is the optimal number (or rather: the **ratio**) of the boundary and outer points?
- Does the two approaches deliver **similar accuracy**?
- With an optimal choice of all parameters, which **convergence rate** can be achieved?
- What is the setup and the parameters in a **neural network** used in the computations?

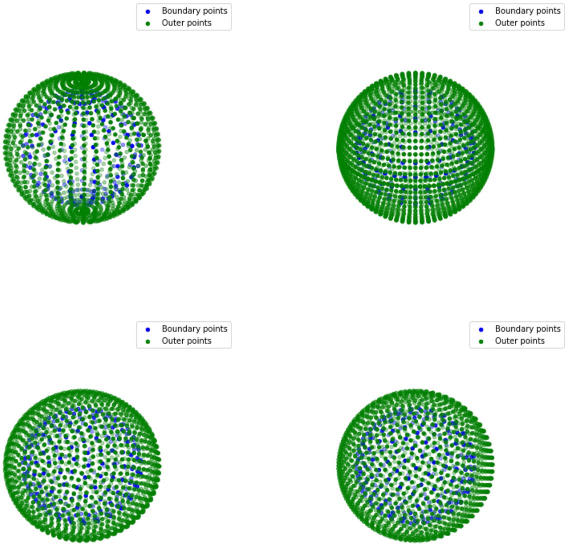
Nodal distribution

Example

$$\begin{cases} \Delta^2 u - 3u = 0 & \text{on } \Omega \\ u(x,y,z) = \sinh(x+y+z) & \text{on } \partial\Omega \end{cases}$$

where $\Omega = \{(x,y,z) | x^2 + y^2 + z^2 < 1\}$.

Nodal distributions



Nodal distributions

Collocation points	Error(dis.1)	Error(dis.2)	Error(dis.3)	Error(dis.4)
$(0.6, 0.6, 0.1)$	-0.007686	0.002031	0.001083	-1.168*e-5
$(0.3, 0.3, 0.3)$	-0.021422	0.005856	0.002753	-0.000319
$(0, 0, 0)$	-0.028143	0.007756	0.003630	-0.000422
$(-.6, .7, -.3)$	-0.000238	0.000237	-0.000335	-0.000581

Table: Different nodal distribution leads to different error distribution

Irregular domains

Example

Let us consider the Laplace equation on the Amoeba-like domain where the boundary points have the position of

$$\left(e^{\sin\theta}\sin^2(2\theta) + e^{\cos\theta}\cos^2(2\theta)\right)(\cos\theta, \sin\theta)\,.$$

The boundary conditions are given by the Dirichlet boundary condition if $0 \le \theta < \pi$ and the Neumann boundary condition if that $0 \le \theta < 2\pi$ such that the analytical solution is

$$u(x, y) = \cos(x)\cosh(y) + \sin(x)\sinh(y)$$

Irregular domains

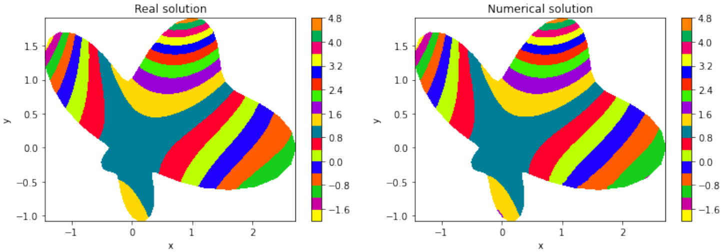


Figure: The analytical and numerical solution on Amoeba-like domain

Collocation points	$(0,0)$	$(1,1)$	$(2,0)$	$(-1,1)$
Analytical solution	1	1.822627	-0.416146	-0.155167
Numerical solution	1.0056722	1.823955	-0.437396	-0.15932

Table: The numerical solution of some points on the mixed boundary problem using the second approach.

Numerical results

Example

In a unit square Ω let us consider Laplace equation with Dirichlet boundary

$$\begin{cases} u(x,0)=0,\ u(x,1)=\sin(\pi x) & \text{for } 0<x<1 \\ u(0,y)=0,\ u(1,y)=0 & \text{for } 0<y<1. \end{cases} \tag{4}$$

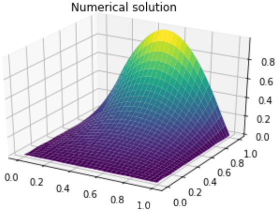


Figure: Numerical solutions are solved by the first approach (left) and the second approach (right) of example on the square

Distance between two surfaces

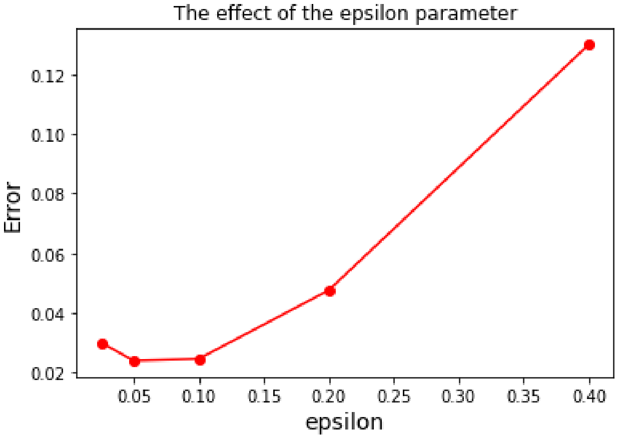


Figure: The effect of ϵ (the first approach)

The number of boundary and outer points

The number of boundary and outer points	Error (Frobenius norm)
$n = 16, m = 64$	0.2461622809658327
$n = 32, m = 64$	0.3182853004347026
$n = 32, m = 128$	0.21974143048835057
$n = 16, m = 128$	0.18014076148115063
$n = 16, m = 512$	0.10919759728510069
$n = 16, m = 2048$	0.07990843843706859
$n = 32, m = 8192$	0.04762830527750495

Table: The effect of the number of boundary points and outer points (the first approach)

Collocation points	True solution	(100,100)	$(400, 400)$	$(1000, 100)$	$(100, 1000)$
$(0.2, 0.2)$	0.034124	0.031156	0.033824	0.0342969	0.0341972
$(0.5, 0.5)$	0.199268	0.200808	0.200867	0.1984138	0.1958812
$(0.7, 0.7)$	0.311947	0.314457	0.311565	0.311674	0.306193
$(0.2, 0.7)$	0.226643	0.224749	0.223706	0.2267211	0.222510
$(0.7, 0.2)$	0.046969	0.045743	0.046496	0.0466224	0.0464201

Table: The different choices of the number of boundary points and outer points, $\epsilon = 0.15$(the second approach)

Neural Networks Structures

Neural network	Error
Linear Regression, no hidden layer, epochs = 1000	0.048206
Neural network, 1 hidden layer of size 100, epochs = 1000	0.026957
Neural network, 1 hidden layer of size 500, epochs = 1000	0.026286
Neural network, 2 hidden layers of size 100, epochs = 1000	0.022334
CNN including convolution of 2 FC layers, epochs = 10000	0.017556

Table: Different neural network structures for example on the square.(the first approach)

Future works

- If the fundamental solution is not continuous? (E.g. The wave equation)
- If the equations do not have fundamental solutions?
- Error estimation on Sobolev space.
- Application on the acoustic problem.
- Application on moving boundary problem.

THANK YOU FOR YOUR ATTENTION!