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Numerical solution of a nonlinear plate equation

Directed Studies 2

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Budapest, 2020

1.The gradient method in Sobolev space:

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain
$\langle f,g\rangle_{L^2(\Omega)} := \int_\Omega fg \ (f,g \in L^2(\Omega))$ and let $H := \left(L^2(\Omega), \langle .,. \rangle_{L^2(\Omega)}\right)$.
We define a differential operator T with domain
$dom\, T := D := \{u \epsilon H^4(\Omega) : u_{|\partial\Omega} = \frac{\partial^2 u}{\partial v^2}|_{\partial\Omega} = 0\}$

$$T(u) := div^2\left(\overline{g}\left(E(D^2u)\right)\widetilde{D}^2 u\right) \quad (u\epsilon D)$$

where $E(D^2u) = \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$,

$$\widetilde{D}^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} & \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} \\ \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \end{pmatrix}$$

Let $\overline{g}$ be real function that is C^2 in the variable r, and there exist $M, m, \lambda > 0$ such that

$$0 < m \le \overline{g}(r) \le M \qquad (r \ge 0)$$

$$0 < m \le (\overline{g}(r^2)r)' \le M \quad (r \ge 0)$$

$$\left|\frac{\partial^2}{\partial r^2}(\overline{g}(r^2)r\right| \le \lambda \qquad (r \ge 0)$$

Finally let $B := \Delta^2, domB := D$
(Throughout the report $(\Delta)^2$ is considered as an $L^2(\Omega) \to L^2(\Omega)$ operator with domain D.)
The following boundary value problem will be considered :

$$\begin{cases} T(u) = \alpha \\ u_{|\partial\Omega} = \frac{\partial^2 u}{\partial v^2}|_{\partial\Omega} = 0 \end{cases} \qquad (1)$$

where $\alpha \epsilon L^2(\Omega)$ is given and u is the unknown function .

The equation (1) describes the elasto-plastic bending of a freely supported thin plane $\Omega \subset \mathbb{R}^2$ under vertical force .

Theorem 1 (see [1]) : Consider the boundary value problem:

$$\begin{cases} T(u) = \alpha \\ u_{|\partial\Omega} = \frac{\partial^2 u}{\partial v^2}|_{\partial\Omega} = 0 \end{cases} \qquad (2)$$

where T is defined as above, let $\alpha \in L^2(\Omega)$ be arbitrary. Then problem (2) admits a unique weak solution $u^* \in H^2(\Omega) \cap H_0^1(\Omega)$,that is for all $v \in H_0^2(\Omega)$

$$\frac{1}{2}\int_{\Omega} \overline{g}(E(D^2u^*))(D^2u^*.D^2v + \Delta u^*\Delta v) = \int_{\Omega} \alpha v$$

holds. (If $u^* \epsilon D$,then $T(u^*) = \alpha$)

The operator F, defined by $\langle F(u), v\rangle$ on the left hand side satisfies

$$m\|h\|^2_{H_0^2(\Omega)} \leq \langle F'(u)h, h\rangle_{H_0^2(\Omega)} \leq M\|h\|^2_{H_0^2(\Omega)} \quad (3)$$

Theorem 2 : (see [1]) Let $u_0 \epsilon D$ be arbitrary. Then the following sequence :

$$u_{n+1} := u_n - \frac{2}{M+m} z_n \quad (n \epsilon \mathbb{N})$$

where
$$\begin{cases} \Delta^2 z_n = T(u_n) - \alpha \\ z_{|\partial\Omega} = \frac{\partial^2 z}{\partial v^2}|_{\partial\Omega} = 0 \end{cases} \quad (4)$$

converges to the solution u^* and

$$\|u - u^*\|_{H_0^2(\Omega)} \leq \frac{1}{m\sqrt{\lambda_1}} \|T(u_0) - g\|_{L^2(\Omega)}. (\frac{M-m}{M+m})^n,$$

where $\lambda_1 > 0$ is the smallest eigenvalue of (Δ^2) on D.

Remark : (a) Weak form of (4) :

$$\int_{\Omega} D^2 z_n . D^2 v = \frac{1}{2}\int_{\Omega} \overline{g}(E(D^2u))(D^2u.D^2v + \Delta u\Delta v) - \int_{\Omega} \alpha v \qquad \left(v \in H_0^2(\Omega)\right)$$

(b) From now $\alpha > 0$ is constant .

2.The Gradient-Fourier method :

Let $\Omega = [0,\pi]^2$, $\lambda_{k,l}$ and $e_{k,l}$ $(k,l = 1,2,...)$ denote the eigenvalues and eigenfunctions of (Δ^2) on D, respectively: $\lambda_{k,l} = (k^2 + l^2)^2,\ e_{k,l}(x,y) = \frac{2}{\pi}\sin(kx)\sin(ly)$.

Let us first introduce some notations. For n = 0,1, … , let

$$\overline{u_0} = u_0$$

$$\overline{u}_{n+1} := \overline{u_n} - \frac{2}{M+m}\overline{z_n} \quad (n\epsilon\mathbb{N})$$

where $\overline{z_n}$ is the solution to the auxiliary equation (4) obtained by the Fourier method

3.1. The Fourier method for the auxiliary equations :

Now let us focus on a single iteration step (i.e. n $\epsilon\mathbb{N}$ is fixed in the section), where (4) is replaced by

$$\begin{cases} \Delta^2\overline{z_n} = \overline{r_n} \\ \overline{z_n}_{|\partial\Omega} = \frac{\partial^2\overline{z_n}}{\partial v^2}|_{\partial\Omega} = 0 \end{cases} \quad (5)$$

which now can be solved by a formula

Let $c_{k,l}(k,l = 1,2,...)$ be the coefficients of $\overline{r_n}$ in its Fourier series expansion, that is

$$c_{k,l} := \int_{\Omega} \overline{r_n}\, e_{k,l}$$

Let N be a positive fixed integer and

$$\overline{r_n} := \sum_{k,l=1}^{N} c_{k,l}e_{k,l}$$

Define $\overline{z_n}$,as $\overline{z_n} := \sum_{k,l=1}^{N} \frac{c_{k,l}}{\lambda_{k,l}} e_{k,l}$

A simple calculation shows that these satisfy (5) :

$$\Delta^2 \overline{z_n} = \sum_{k,l=1}^{N} \frac{c_{k,l}}{\lambda_{k,l}} (\Delta^2 e_{k,l}) = \sum_{k=1}^{N} \frac{c_{k,l}}{\lambda_{k,l}} \lambda_{k,l} e_{k,l} = \sum_{k,l=1}^{N} c_{k,l} e_{k,l} = \overline{r_n}$$

3.2 Calculation of the coefficients $c_{k,l}$: Let $u := u_n$

$$c_{k,l} := \int_{\Omega} (T(u) - \alpha) e_{k,l}$$

$$:= \int_{\Omega} div^2 \left(\overline{g} \left(E(D^2 u) \right) \widetilde{D}^2 u \right)^* e_{k,l} - \alpha * e_{k,l}$$

$$:= \int_{\Omega} \bar{g}(\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2) * (\widetilde{D}^2 u . \widetilde{D}^2 e_{k,l}) - \int_{\Omega} \alpha * e_{k,l}$$

$$J := \frac{\partial^2 u}{\partial x^2} = \sum_{k,l=1}^{N} d_{k,l} \frac{\partial^2 e_{k,l}}{\partial x^2} = - \sum_{k,l=1}^{N} \frac{2}{\pi} d_{k,l}\, k^2 \sin(kx) \sin(ly)$$

$$E := \frac{\partial^2 u}{\partial y^2} = \sum_{k,l=1}^{N} d_{k,l} \frac{\partial^2 e_{k,l}}{\partial y^2} = - \sum_{k,l=1}^{N} \frac{2}{\pi} d_{k,l}\, l^2 \sin(kx) \sin(ly)$$

$$G := \frac{\partial^2 u}{\partial x \partial y} = \sum_{k,l=1}^{N} d_{k,l} \frac{\partial^2 e_{k,l}}{\partial x \partial y} = \sum_{k,l=1}^{N} \frac{2}{\pi} d_{k,l}\, kl \cos(kx) \cos(ly)$$

$$Q := \frac{\partial^2 e_{k,l}}{\partial x^2} = - \frac{2}{\pi} k^2 \sin(kx) \sin(ly)$$

$$M := \frac{\partial^2 e_{k,l}}{\partial y^2} = - \frac{2}{\pi} l^2 \sin(kx) \sin(ly)$$

$$K := \frac{\partial^2 e_{k,l}}{\partial x \partial y} = \frac{2}{\pi} kl * \cos(kx)\cos(ly)$$

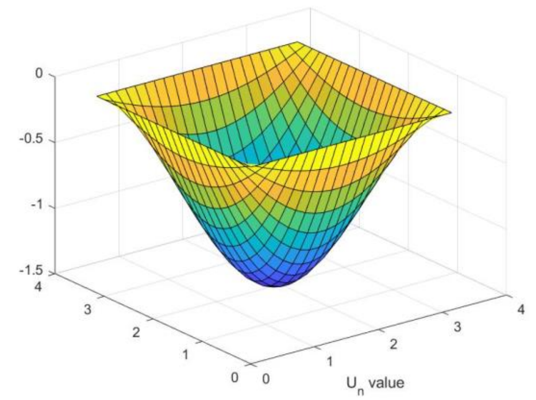
$$c_{k,l} = \int_{\Omega} \bar{g}((J)^2 + J * E + (E)^2 + (G)^2) * \left(\left(J + \frac{1}{2}E\right) * Q + G * K + \left(E + \frac{1}{2}J\right) * M\right) - \alpha e_{k,l}$$

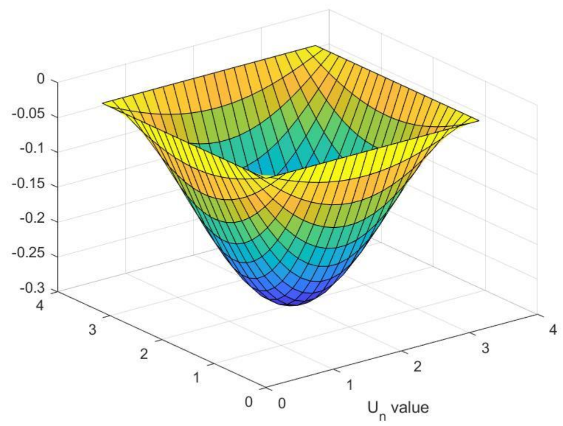
where the $\bar{g}(t) := \frac{1}{1+\sqrt{1-\frac{t}{3}}}$, then m=0,51 and M=2,81 (see [2])

Convergence :Apply Theorem 2 for the iteration in the Galerkin subspace $V_h = span\ \{e_{k,l}\}_{k,l=1,\dots,N}$

Conclusions :

We have run the iterative algorithm for the plate model with constant force and for N=15. As examples, the first figure is the iterate n=3 (left graph) and the other is for n=6 (right graph), which show that the graphs are the shape of the deformed plate under the vertical constant force $\alpha > 0$.





References :

[1] I. Faragó, J. Karátson. Numerical solution of nonlinear elliptic problems via preconditioning operators, Nova Science Publ., 2002.

[2] L. Loczi, The gradient-Fourier method for nonlinear elliptic partial differential equations in Sobolev space, Ann. Univ. Sci. Budapest. 43 (2000), 139–149 (2001).

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