# An improved Kalai-Kleitman bound for the diameter of a polyhedron

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**Abstract**

Kalai and Kleitman [1] established the bound $n^{\log(d)+2}$ for the diameter of a $d$-dimensional polyhedron with $n$ facets. Here we improve the bound slightly to $n^{\log(d)}$.

## 1 Introduction

A $d$-polyhedron $P$ is a $d$-dimensional set in $\mathbb{R}^d$ that is the intersection of a finite number of half-spaces of the form $H := \{x \in \mathbb{R}^d : a^T x \leq \beta\}$. If $P$ can be written as the intersection of $n$ half-spaces $H_i$, $i = 1, \ldots, n$, but not fewer, we say it has $n$ facets and these facets are the faces $F_i = P \cap H_i$, $i = 1, \ldots, n$, each linearly isomorphic to a $(d-1)$-polyhedron with at most $n-1$ facets. We then call $P$ a $(d, n)$-polyhedron.

We say $v \in P$ is a *vertex* of $P$ if there is a half-space $H$ with $P \cap H = \{v\}$. Two vertices $v$ and $w$ of $P$ are *adjacent* (and the set $[v, w] := \{(1-\lambda)v + \lambda w : 0 \leq \lambda \leq 1\}$ an *edge* of $P$) if there is a half-space $H$ with $P \cap H = [v, w]$. A *path of length* $k$ from vertex $v$ to vertex $w$ in $P$ is a sequence $v = v_0, v_1, \ldots, v_k = w$ of vertices with $v_{i-1}$ and $v_i$ adjacent for $i = 1, \ldots, k$. The *distance* from $v$ to $w$ is the length of the shortest such path and is denoted $d_P(v, w)$, and the diameter of $P$ is the largest such distance,

$$\delta(P) := \max\{d_P(v, w) : v \text{ and } w \text{ vertices of } P\}.$$

We define

$$\Delta(d, n) := \max\{\delta(P) : P \text{ a } (d, n)\text{-polyhedron}\}$$

and seek an upper bound on $\Delta(d, n)$. It is not hard to see that $\Delta(d, \cdot)$ is monotonically non-decreasing. Also, the maximum above can be attained by a *simple* polyhedron, one where each vertex lies in exactly $d$ facets. See, e.g., Klee and Kleinschmidt [2] or Ziegler [3]. A related paper, Ziegler [4], gives the history of the Hirsch conjecture on $\Delta_b(d, n)$, defined as above but for bounded polyhedra.

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## 2 Result

We prove

**Theorem 1** *For $1 \le d \le n$, $\Delta(d,n) \le d^{\log(n)}$.*

(All logarithms are to base 2; note that $d^{\log(n)} = n^{\log(d)}$ as both have logarithm $\log(d) \cdot \log(n)$. We use this in the proof below.)

The key lemma is due to Kalai and Kleitman [1], and was used by them to prove the bound $n^{\log(d)+2}$. We give the proof for completeness.

**Lemma 1** *For $2 \le d \le \lfloor n/2 \rfloor$, where $\lfloor n/2 \rfloor$ is the largest integer at most $n/2$,*

$$\Delta(d,n) \le \Delta(d-1,n-1) + 2\Delta(d,\lfloor n/2 \rfloor) + 2.$$

**Proof:** Let $P$ be a simple $(d,n)$-polyhedron and $v$ and $w$ two vertices of $P$ with $\delta_P(v,w) = \Delta(d,n)$. We show there is a path in $P$ from $v$ to $w$ of length at most the right-hand side above. If $v$ and $w$ both lie on the same facet, say $F$, of $P$, then since $F$ is linearly isomorphic to a $(d-1,m)$-polyhedron with $m \le n-1$, we have $d_P(v,w) \le d_F(v,w) \le \Delta(d-1,m) \le \Delta(d-1,n-1)$ and we are done.

Otherwise, let $k_v$ be the largest $k$ so that there is a set $\mathcal{F}_v$ of at most $\lfloor n/2 \rfloor$ facets with all paths of length $k$ from $v$ meeting only facets in $\mathcal{F}_v$. This exists since all paths of length 0 meet only $d$ facets (those containing $v$), whereas paths of length $\delta(P)$ can meet all $n$ facets of $P$. Define $k_w$ and $\mathcal{F}_w$ similarly. We claim that $k_v \le \Delta(d,\lfloor n/2 \rfloor)$ and similarly for $k_w$. Indeed, let $P_v \supseteq P$ be the $(d,m_v)$-polyhedron ($m_v = |\mathcal{F}_v| \le \lfloor n/2 \rfloor$) defined by just those linear inequalities corresponding to the facets in $\mathcal{F}_v$. Consider any vertex $t$ of $P$ a distance $k_v$ from $v$, so there is a shortest path from $v$ to $t$ of length $k_v$ meeting only facets in $\mathcal{F}_v$. But this is also a shortest path in $P_v$, since if there were a shorter path, it could not be a path in $P$, and thus must meet a facet not in $\mathcal{F}_v$, a contradiction. So

$$k_v = \delta_{P_v}(v,t) \le \Delta(d,m_v) \le \Delta(d,\lfloor n/2 \rfloor).$$

Now consider the set $\mathcal{G}_v$ of facets that can be reached in at most $k_v+1$ steps from $v$, and similarly $\mathcal{G}_w$. Since both these sets contain more than $\lfloor n/2 \rfloor$ facets, there must be a facet, say $G$, in both of them. Thus there are vertices $t$ and $u$ in $G$ and paths of length at most $k_v+1$ from $v$ to $t$ and of length at most $k_w+1$ from $w$ to $u$. Then

$$\begin{aligned}
\Delta(d,n) &= d_P(v,w) \\
&\le d_P(v,t) + d_G(t,u) + d_P(w,u) \\
&\le k_v + 1 + \Delta(d-1,n-1) + k_w + 1 \\
&\le \Delta(d-1,n-1) + 2\Delta(d,\lfloor n/2 \rfloor) + 2,
\end{aligned}$$

since, as above, $G$ is linearly isomorphic to a $(d-1,m)$-polyhedron with $m \le n-1$. □

**Proof of the theorem:** This is by induction on $d+n$. First, the right-hand side gives 1 for $d=1$, which is clearly a valid bound, and $n$ for $d=2$ which is an upper bound on the true value of $n-2$.

For $d=3$, if $n<6$ any two vertices lie on a common facet, so their distance is at most $\Delta(2,n-1) = n-3 < n^{\log(3)}$. If $n \ge 6$, we use the lemma to obtain

$$\begin{aligned}
\Delta(3,n) &\le \Delta(2,n-1) + 2 \cdot 3^{\log(\lfloor n/2 \rfloor)} + 2 \\
&\le n - 3 + 2 \cdot 3^{\log(n)-1} + 2 \\
&= n - 1 + \frac{2}{3} \cdot 3^{\log(n)} = n - 1 + \frac{2}{3} \cdot n^{\log(3)}.
\end{aligned}$$

Thus it suffices to show $n-1 \leq \frac{1}{3}n^{\log(3)}$ for $n \geq 6$, and this can be confirmed by looking at the values at $n=6$ and the derivatives for $n \geq 6$. (In fact, $\Delta(3,n) = n-3$; see [2]. We have chosen to give a self-contained argument.)

For $d \geq 4$ and $n < 2d$, the result will follow by induction since any two vertices lie on a common facet giving $\Delta(d,n) \leq \Delta(d-1,n-1)$. For $d=4$ and $n=8$, the distance between any two vertices lying on a common facet will be at most $\Delta(3,7)$ as above, while if $v$ and $w$ lie on disjoint facets, any (bounded) edge from $v$ leads to a vertex $u$ on a common facet with $w$, so the distance is at most $1+\Delta(3,7)$, which again suffices. The only remaining case is $d \geq 4$, $n \geq 9$. For this, $\log(n-1) \geq 3$, so we have

$$\begin{aligned}
\Delta(d,n) &\leq \Delta(d-1,n-1) + 2\cdot\Delta(d,\lfloor n/2\rfloor) + 2 \\
&\leq (d-1)^{\log(n-1)} + 2\cdot d^{\log(n)-1} + 2 \\
&= \left(\frac{d-1}{d}\right)^{\log(n-1)} d^{\log(n-1)} + \frac{2}{d}\cdot d^{\log(n)} + 2 \\
&\leq \left(\frac{d-1}{d}\right)^{3} d^{\log(n)} + \frac{2}{d}\cdot d^{\log(n)} + 2 \\
&\leq d^{\log(n)} - \frac{3}{d}\cdot d^{\log(n)} + \frac{3}{d^2}\cdot d^{\log(n)} - \frac{1}{d^3}\cdot d^{\log(n)} + \frac{2}{d}\cdot d^{\log(n)} + 2 \\
&= d^{\log(n)} - \frac{1}{d}\cdot d^{\log(n)} + \frac{3}{d^2}\cdot d^{\log(n)} - \frac{1}{d^3}\cdot d^{\log(n)} + 2 \\
&\leq d^{\log(n)} - \frac{1}{d}\cdot d^{\log(n)} + \frac{3}{4d}\cdot d^{\log(n)} - \frac{1}{d^3}\cdot d^{\log(n)} + 2 \\
&\leq d^{\log(n)} - \frac{1}{4d}\cdot d^{\log(n)} - \frac{1}{d^3}\cdot d^{\log(n)} + 2 \\
&\leq d^{\log(n)},
\end{aligned}$$

since each of the subtracted terms is at least one. This completes the proof. □

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# References

[1] G. Kalai and D.J. Kleitman, A quasi-polynomial bound for the diameter of graphs of polyhedra, *Bulletin of the American Mathematical Society* 26 (1992), 315–316.

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[4] G. Ziegler, Who solved the Hirsch conjecture? *Documenta Mathematica* Extra Volume: Optimization Stories (2012), 75–85.