AN IMPROVED KALAI-KLEITMAN BOUND FOR THE DIAMETER OF A POLYHEDRON

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Abstract. Kalai and Kleitman [6] established the bound $n^{\log(d)+2}$ for the diameter of a d-dimensional polyhedron with n facets. Here we improve the bound slightly to $(n-d)^{\log(d)}$.

1. Introduction.

A d-polyhedron P is a d-dimensional set in $\mathbb{R}^d$ that is the intersection of a finite number of half-spaces of the form $H := \{x \in \mathbb{R}^d : a^T x \leq \beta\}$. If P can be written as the intersection of n half-spaces H_i (with bounding hyperplanes H_i^0), $i = 1, \ldots, n$, but not fewer, we say it has n facets and these facets are the faces $F_i = P \cap H_i^0$, $i = 1, \ldots, n$, each affinely isomorphic to a $(d-1)$-polyhedron with at most $n-1$ facets. We then call P a (d, n)-polyhedron.

We say $v \in P$ is a *vertex* of P if there is a half-space H with $P \cap H = \{v\}$. (A polyhedron is *pointed* if it has a vertex, or equivalently, if it contains no line.) Two vertices v and w of P are *adjacent* (and the set $[v, w] := \{(1-\lambda)v + \lambda w : 0 \leq \lambda \leq 1\}$ an *edge* of P) if there is a half-space H with $P \cap H = [v, w]$. A *path of length* k from vertex v to vertex w in P is a sequence $v = v_0, v_1, \ldots, v_k = w$ of vertices with v_{i-1} and v_i adjacent for $i = 1, \ldots, k$. The *distance* from v to w is the length of the shortest such path and is denoted $\rho_P(v, w)$, and the diameter of P is the largest such distance,

$$\delta(P) := \max\{\rho_P(v, w) : v \text{ and } w \text{ vertices of } P\}.$$

We define

$$\Delta(d, n) := \max\{\delta(P) : P \text{ a } (d, n)\text{-polyhedron}\}$$

and seek an upper bound on $\Delta(d, n)$. It is not hard to see that $\Delta(d, \cdot)$ is monotonically non-decreasing. Also, the maximum above can be attained by a *simple* polyhedron, one where each vertex lies in exactly d facets. See, e.g., Klee and Kleinschmidt [12] or Ziegler [18].

In the last few years, there has been an explosion of papers related to the diameters of polyhedra and related set systems. Much of this was inspired by Santos [14] finding a counterexample to the Hirsch conjecture that $\Delta_b(d, n) \leq n - d$, where $\Delta_b(d, n)$ is defined as above but for bounded polyhedra; see also the refined counterexamples of Matschke, Santos, and Weibel [13]. Eisenbrand, Hähnle, Razborov, and Rothvoss [3] showed that a slightly improved Kalai-Kleitman bound, $n^{\log(d)+1}$, held for a very general class of set families called *base abstractions* which assume only basic combinatorial properties of the vertices of d-polyhedra with n facets. These and other combinatorial abstractions consist of a collection of d-subsets of an n-set, corresponding to the sets of facets containing each vertex of a polyhedron, possibly with additional combinatorial structures; when we wish to highlight the dimensions, we add the prefix (d, n). Base abstractions include the ultraconnected set families considered by Kalai [4] and the abstract polytopes of Adler and Dantzig [1]; the latter satisfy the Hirsch conjecture for $n - d \leq 5$. (The bound $n^{\log(d)+1}$, for polyhedra, was presented first in Kalai [5].) Another class of set families, *subset partition graphs*, was introduced by Kim [7]; adding various properties gave families for which the bound held, or other families where the maximum diameter grew exponentially. This exponential lower bound is due to Santos [15]. Another early combinatorial abstraction of polytopes consists of the duoids of [16, 17]. We also mention the nice overview articles of Kim and Santos [8] (pre-counterexample) and De Loera [2], Santos [15], and Ziegler [18] (post-counterexample).

Our bound $(n-d)^{\log(d)}$ fits better with the Hirsch conjecture and is tight for dimensions 1 and 2. Also, more importantly, it is invariant under linear programming duality. A pointed d-polyhedron with n facets can be written as $\{x \in \mathbb{R}^d : Ax \leq b\}$ for some $n \times d$ matrix A of full rank and some n-vector b. Choosing an objective function $c^T x$ for $c \in \mathbb{R}^d$ gives the linear programming problem $\max\{c^T x : Ax \leq b\}$, whose dual is $\min\{b^T y : A^T y = c,\ y \geq 0\}$. The feasible region for the latter is affinely isomorphic to a pointed polyhedron of dimension at most $n - d$ with at most n facets, and equality is possible. Hence duality switches the dimensions d and $n - d$.

Our proof of the improved bound uses the same lemma as employed by Kalai and Kleitman, with a slightly tighter analysis of the inductive step and the consideration of a number of low-dimensional cases. Hence the bound also applies to combinatorial set systems generalizing polyhedra as long as these ingredients remain valid. While the lemma holds for the base abstractions of Eisenbrand et al. (see the proof of Theorem 3.1 in [3]), that paper also contains in Section 2 a $(2, 6)$-base abstraction with diameter 5, and so the improved bound fails. Similarly, Theorem 7.1 of [16] shows that the diameters of $(3, n)$-duoids grow at least quadratically with n, showing that this combinatorial abstraction also fails to satisfy the improved bound. We do not know the situation for ultraconnected set families or abstract polytopes, both of which satisfy the lemma.

2. Result.

We prove

THEOREM 2.1. *For $1 \leq d \leq n$, $\Delta(d, n) \leq (n-d)^{\log(d)}$, with $\Delta(1, 1) = 0$.*
(All logarithms are to base 2; note that $(n-d)^{\log(d)} = d^{\log(n-d)}$, for $1 < d < n$, as both have logarithm $\log(d) \cdot \log(n-d)$. We use this in the proof below.)

The key lemma is due to Kalai and Kleitman [6], and was used by them to prove the bound $n^{\log(d)+2}$. We give the proof for completeness.

LEMMA 2.2. *For $2 \leq d \leq \lfloor n/2 \rfloor$, where $\lfloor n/2 \rfloor$ is the largest integer at most $n/2$,*

$$\Delta(d, n) \leq \Delta(d-1, n-1) + 2\Delta(d, \lfloor n/2 \rfloor) + 2.$$

Proof. Let P be a simple (d, n)-polyhedron and v and w two vertices of P with $\delta_P(v, w) = \Delta(d, n)$. We show there is a path in P from v to w of length at most the right-hand side above. If v and w both lie on the same facet, say F, of P, then since F is

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affinely isomorphic to a $(d-1,m)$-polyhedron with $m \le n-1$, we have $\rho_P(v,w) \le \rho_F(v,w) \le \Delta(d-1,m) \le \Delta(d-1,n-1)$ and we are done.

Otherwise, let k_v be the largest k so that there is a set $\mathcal{F}_v$ of at most $\lfloor n/2 \rfloor$ facets with all paths of length k from v meeting only facets in $\mathcal{F}_v$. This exists since all paths of length 0 meet only d facets (those containing v), whereas paths of length $\delta(P)$ can meet all n facets of P. Define k_w and $\mathcal{F}_w$ similarly. We claim that $k_v \le \Delta(d, \lfloor n/2 \rfloor)$ and similarly for k_w. Indeed, let $P_v \supseteq P$ be the (d, m_v)-polyhedron ($m_v = |\mathcal{F}_v| \le \lfloor n/2 \rfloor$) defined by just those linear inequalities corresponding to the facets in $\mathcal{F}_v$. Consider any vertex t of P a distance k_v from v, so there is a shortest path from v to t of length k_v meeting only facets in $\mathcal{F}_v$. But this is also a shortest path in P_v, since if there were a shorter path, it could not be a path in P, and thus must meet a facet not in $\mathcal{F}_v$, a contradiction. So

$$k_v = \delta_{P_v}(v,t) \le \Delta(d, m_v) \le \Delta(d, \lfloor n/2 \rfloor).$$

Now consider the set $\mathcal{G}_v$ of facets that can be reached in at most $k_v + 1$ steps from v, and similarly $\mathcal{G}_w$. Since both these sets contain more than $\lfloor n/2 \rfloor$ facets, there must be a facet, say G, in both of them. Thus there are vertices t and u in G and paths of length at most k_v+1 from v to t and of length at most k_w+1 from w to u. Then

$$\begin{aligned}\Delta(d,n) &= \rho_P(v,w) \\ &\le \rho_P(v,t) + \rho_G(t,u) + \rho_P(w,u) \\ &\le k_v + 1 + \Delta(d-1,n-1) + k_w + 1 \\ &\le \Delta(d-1,n-1) + 2\Delta(d, \lfloor n/2 \rfloor) + 2,\end{aligned}$$

since, as above, G is affinely isomorphic to a $(d-1,m)$-polyhedron with $m \le n-1$. ▯ □

Proof of the theorem: This is by induction on $d+n$. The result is trivial for $n=d$, since there can be only one vertex. Next, the right-hand side gives 1 for $d=1$ ($n=2$) and $n-2$ for $d=2$, which are the correct values. For $d=3$, it gives $(n-3)^{\log(3)}$, which is greater than the correct value $n-3$ established by Klee [9, 10, 11]. (We could make the proof more self-contained by establishing the $d=3$ case from the lemma: a general argument deals with $n \ge 13$, but then there are seven more special cases to check.) Below we will give a general inductive step for the case $d \ge 4$, $n-d \ge 8$. Also, the result clearly holds by induction if $n < 2d$, since then any two vertices lie on a common facet, so their distance is at most $\Delta(d-1,n-1)$. The remaining cases are $d=4$, $8 \le n \le 11$; $d=5$, $10 \le n \le 12$; $d=6$, $12 \le n \le 13$; and $d=7$, $n=14$. All these cases can be checked easily using the lemma, the equation $\Delta(d,d)=0$, and the equations $\Delta(5,6) = \Delta(4,5) = \Delta(3,4) = \Delta(2,3) = 1$.

Now we deal with the case $d \ge 4$, $n-d \ge 8$. For this, $\log(n-d) \ge 3$, so we have

$$\begin{aligned}\Delta(d,n) &\le \Delta(d-1,n-1) + 2 \cdot \Delta(d, \lfloor n/2 \rfloor) + 2 \\ &\le (d-1)^{\log(n-d)} + 2 \cdot d^{\log(n/2-d)} + 2 \\ &\le \left(\frac{d-1}{d}\right)^{\log(n-d)} d^{\log(n-d)} + 2 \cdot d^{\log((n-d)/2)} + 2 \\ &\le \left(\frac{d-1}{d}\right)^{3} d^{\log(n-d)} + \frac{2}{d} \cdot d^{\log(n-d)} + 2 \\ &= \left(1 - \frac{3}{d} + \frac{3}{d^2} - \frac{1}{d^3} + \frac{2}{d}\right) d^{\log(n-d)} + 2 \\ &\le \left(1 - \frac{1}{d} + \frac{3}{4d} - \frac{1}{d^3}\right) d^{\log(n-d)} + 2 \\ &\le d^{\log(n-d)} - \frac{1}{4d} \cdot d^{\log(n-d)} - \frac{1}{d^3} \cdot d^{\log(n-d)} + 2 \\ &\le d^{\log(n-d)},\end{aligned}$$

since each of the subtracted terms is at least one. This completes the proof. □

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