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Numerical solution of a nonlinear plate equation

Directed Studies 1

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Abstract :

In this report we will discuss in the first part the PDE model which is the equation of elasto-plastic bending of clamped plates and try to understand the weak form of this equation.We will prove that it has a unique weak solution.

The second part is about theory for the numerical solution, it revolves around construction and proof of convergence, It consists of finite element discretization and inner-outer iterations.

1.1 The PDE :

Elasto-plastic bending of a clamped thin plane plate $\Omega \in \mathbb{R}^2$ is described by a fourth order nonlinear Dirichlet boundary value problem .

The formulation of the problem is the following :

$$\begin{cases} \frac{\partial^2}{\partial x^2}\left(\overline{g}(E(D^2u))\left(\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}\right)\right)+\frac{\partial^2}{\partial x\partial y}\left(\overline{g}(E(D^2u))\left(\frac{\partial^2 u}{\partial x\partial y}\right)\right) \\ \qquad +\frac{\partial^2}{\partial y^2}\left(\overline{g}(E(D^2u))\left(\frac{\partial^2 u}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right)\right) = \alpha(x) \\ u_{|\partial\Omega}=\frac{\partial u}{\partial \nu}|_{\partial\Omega}=0 \end{cases}$$

This problem is written briefly as

$$\begin{cases} Div^2\left(\overline{g}\left(E(D^2u)\right)\widetilde{D}^2u\right)=\alpha(x) \\ u_{|\partial\Omega}=\frac{\partial u}{\partial v}|_{\partial\Omega}=0 \end{cases}$$

where the scalar function $\overline{g} \in C^1(\mathbb{R}^+)$satisfies the condition

$$0<\mu_1 \le \overline{g}(r) \le \mu_2$$
$$0<\mu_1 \le (\overline{g}(r^2)r)' \le \mu_2$$

with suitable constants $\mu_1, \mu_2 > 0$ independent of the variable $r > 0$.

$$E(D^2u)=\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial^2 u}{\partial y^2}\right)^2+\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2$$

where $D^2u=\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 u}{\partial x\partial y} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix}$

1.2 <u>The weak formulation of problem :</u>

The weak formulation of the problem : find $u \in H_0^2(\Omega)$ such that

$$\frac{1}{2}\int_{\Omega} \overline{g}(E(D^2u))(D^2u.D^2v + \Delta\text{u}\Delta\text{v}) = \int_{\Omega} \alpha v \qquad \left(v \in H_0^2(\Omega)\right) \qquad (1.2.1)$$

For regular functions $u \in H^4(\Omega) \cap H_0^2(\Omega)$, the weak formulation is obtained via multiplying our problem by $v \in H_0^2(\Omega)$, integration and the divergence theorem. In this way we have

$$\int_{\Omega} \overline{g}(E(D^2u))\widetilde{D}^2u.D^2v = \int_{\Omega} \alpha v \qquad \left(v \in H_0^2(\Omega)\right) \qquad (1.2.2)$$

Instead of (1) .The latter can be obtained form here by defining

$$\widetilde{D}^2u = \frac{1}{2}(D^2u. \; + \Delta\text{u}.I_{2\times2})$$

And $I_{2\times2}.D^2v = \Delta\text{v}$, which yields that

$$\widetilde{D}^2u.D^2v = \frac{1}{2}(D^2u.D^2v + \Delta\text{u}\Delta\text{v})$$

2. <u>Prove the existence and uniqueness of the weak solution :</u>

2.1 <u>Theorem:</u>

Let H be a real Hilbert space and let the operator $F: H \to \mathrm{H}$ have the following properties:

(i) F has a bihemicontinuous symmetric Gateaux derivative

(ii) there exists a constant $m > 0$ such that

$$\langle F'(u)h, h\rangle \geq m\|h\|^2 \qquad (u, v \in H) \qquad (2.1.1)$$

Then for any b $\in H$ the equation $F(u) = b$ has a unique solution $u^* \in H$.

2.2 <u>Remark :</u>

Let F have the form :

$$\langle F(u), v\rangle_{H_0^2(\Omega)} = \int_{\Omega} a([u,u])[u,v] \qquad (u, v \in H_0^2(\Omega)^r), \qquad (2.2.1)$$

where the scalar C^1function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the condition

$$0 < \lambda_1 \leq a(r) \leq \lambda_2$$

$$0 < \lambda_1 \leq (a(r^2)r)' \leq \lambda_2$$

Then (2.2.1) defines an operator $F: H_0^2(\Omega) \to H_0^2(\Omega)$ which has a bihemicontinuous symmetric Gateaux derivative satisfying

$$\lambda_1\|h\|^2_{H_0^2(\Omega)} \leq \langle F'(u)h, h\rangle_{H_0^2(\Omega)} \leq \lambda_2\|h\|^2_{H_0^2(\Omega)}$$

$$\lambda_1 \int_{\Omega} [h,h] \leq \left\langle F'^{(u)}h, h\right\rangle_{H_0^2(\Omega)} \leq \lambda_2 \int_{\Omega} [h,h]$$

2.3 Proposition: The elasto-plastic bending problem of a clamped plate has a unique weak solution $u^* \in H_0^2(\Omega)$.

Proof :

For any matrices $B, C \in \mathbb{R}^{2\times2}$let us introduce the following notations:

$\widetilde{B} = \frac{1}{2}(B + trB.I_2)$,$\{B, C\} = \frac{1}{2}(B.C + trB\, trC)$, $E(C) = \{C, C\}$

we verify directly via Remark (2.1) for the operator

$$\langle F(u), v\rangle_{H_0^2} = \int_\Omega f(x, D^2u)\,.D^2v$$

$$= \int_\Omega \overline{g}(E(D^2u))\widetilde{D}^2u.D^2v \qquad \big(u, v \in H_0^2(\Omega)\big)$$

where $f: \Omega \times \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N},\ A(x,\eta) \coloneqq \overline{g}\big(E(\eta)\big).\widetilde{\eta}$

Using the weak form and previously notations , we have

$$\langle F(u), v\rangle_{H_0^2} = \frac{1}{2}\int_\Omega \overline{g}(E(D^2u))(D^2u.D^2v + \Delta u \Delta v)$$

$$= \int_\Omega \overline{g}(\{D^2u\,, D^2u\})\{D^2u.D^2v\} \qquad \big(u, v \in H_0^2(\Omega)\big)$$

The obtained form of F is a special case of (2.2.1) with

$$a(r) = \overline{g}(r) \qquad and\ \ [u, v] = \{D^2u, D^2v\},$$

hence Theorem (2.1) can be applied.

3.Finite element discretization :

3.1 Galerkin's method for nonlinear operator equations :

Let H be a real Hilbert space and $A: H \to H$ a given operator, which is uniformly monotone and Lipschitz continuous. Consider the operator equation $A(u) = b$ (3.1.1)

where $b \in H$. The equation (3.1.1) admits a unique solution $u^* \in H$.

Let us write the equation (3.1.1) in its equivalent variational forms involving test functions :

$$\langle A(u^*), v\rangle = \langle b, v\rangle \quad (\forall v \in H)$$

Let $V_h = span\{\varphi_1, \ldots, \varphi_n\} \subset \mathrm{H}$,The approximate solution $u^h \epsilon\, V_h$ is defined by the subspace equation

$$\langle A(u^h), v^h\rangle = \langle b, v^h\rangle \quad (\forall v^h \in V_h) \qquad (3.1.2)$$

The equation (3.1.2) admits a unique solution $u^h \in V_h$.

The coefficients of the expansion $u^h = \sum_{i=1}^{n} c_i\varphi_i$,can be obtained as follows. We set $v^h \coloneqq \varphi_k$

$$\left\langle A\left(\sum_{i=1}^{n} c_i\varphi_i\right), \varphi_k\right\rangle = \langle b, \varphi_k\rangle \quad \mathrm{k} = (1, \ldots, \mathrm{n})$$

Let us introduce the real functions
$\boldsymbol{\mathcal{A}}_K: \mathbb{R}^n \to \mathbb{R}$ $\boldsymbol{\mathcal{A}}_K(c_1, \dots, c_n) := \langle A(\sum_{i=1}^{n} c_i\varphi_i), \varphi_k\rangle$
and let $\boldsymbol{\mathcal{B}}_k := \langle b, \varphi_k\rangle$ $\mathrm{k} = (1, \dots, \mathrm{n}) \in \mathbb{R}^n$, Now put these functions together in $\boldsymbol{\mathcal{A}}_K: \mathbb{R}^n \to \mathbb{R}^n$ so the coefficients u^h of can be obtained by solving the nonlinear algebraic system of equations $\boldsymbol{\mathcal{A}}_K$ (c) = $\boldsymbol{\mathcal{B}}_k$

3.2 <u>Nonlinear Céa's lemma :</u>

For the Galerkin solution $u^h \in V_h$,the quasi-optimality relation
$\|u^* - u^h\| \le \frac{M}{m}\min\{\|u^* - v^h\|: v^h \in V_h\}$,holds true

3.3 <u>Example :</u> *Consider the problem :* $\begin{cases} div^2 f(x, D^2u) = \alpha(x) \\ u_{|\partial\Omega} = \partial_\nu u_{|\partial\Omega} = 0 \end{cases}$

where $f: \Omega \times \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$, $f(x,\eta) := \overline{g}(E(\eta)).\widetilde{\eta}$ and due to the assumptions on $\overline{g}$ there exists a unique weak solution u^*, that is,

$$\int_\Omega f(x, D^2u^*) . D^2v = \int_\Omega \alpha v \quad \forall v \in H_0^2(\Omega)$$

Let $V_h \in H_0^2(\Omega)$ *be some finite element subspace ,Then the approximate solution ,*$u^h \in V_h$ *satisfies the subspace equation :*

$$\int_\Omega f(x, D^2u^h) . D^2v^h = \int_\Omega \alpha v^h \quad \forall v^h \in V_h$$

and the coefficients can be obtained by solving the nonlinear system of algebraic equations : $\boldsymbol{\mathcal{A}}$ (c) = $\boldsymbol{\mathcal{B}}$

where $\boldsymbol{\mathcal{A}}_k(c) = \int_\Omega f(x, \sum_{i=1}^{n} c_i D^2\varphi_i) . D^2\varphi_K$ *and* $\boldsymbol{\mathcal{B}}_k := \int_\Omega \alpha\varphi_K\ (k = \overline{1,n})$

Here, $\boldsymbol{\mathcal{A}}$ inherits the uniform monotonicity and Lipschitz continuity of f, so unique solvability of this system follows from the theorem on the Galerkin method.Further, the nonlinear Céa's lemma holds true

$$|u^* - u^h|_2 \le \frac{M}{m}\left|u^* - \prod\nolimits_h u^*\right|_2 \le \frac{M}{m} c\, h^{k-1}|u^*|_{k+1} \qquad (u^* \in H_0^{k+1}(\Omega))$$

4 <u>.Inner-outer iterations :</u>

4.1 <u>Asumptions :</u>

Let $f \in C^1(\overline{\Omega} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$ and the Jacobians $\frac{\partial f(x,\eta)}{\partial\eta}$ are symmetric and there are $\mathrm{M} \ge m > 0$,such that :

$$m|\varepsilon|^2 \le \frac{\partial f(x,\eta)}{\partial\eta}\varepsilon.\varepsilon \le M|\varepsilon|^2 \ (x \in \Omega, \varepsilon, \eta \in \mathbb{R}^{n\times n})$$

hold. Let $V \in H_0^2(\Omega)$ be a finite dimensional subspace with the inner product and let $F: V \to \mathrm{V}$

$$\langle F(u), v\rangle_{H_0^2} = \int_\Omega f(x, D^2u). D^2v \qquad (v \in V)$$

and $b \in V$ $\qquad \langle b, v\rangle_{H_0^2} = \int_\Omega gv \qquad (v \in V)$

Denote by $u^* \in V$ the solution of : $\langle F(u^*), v\rangle_{H_0^2} = \langle b, v\rangle_{H_0^2}$

The operator F is Gâteaux differentiable and its derivative is given by

$$\langle F'(u)v, z\rangle_{H_0^2} = \int_\Omega \frac{\partial f}{\partial\theta}(x, D^2u)D^2v . D^2z \qquad (u, v, z \in V)$$

The operator F' inherits the Lipschitz continuity of $\frac{\partial f}{\partial \theta}$, Let L denote the Lipschitz constant of F' .

4.2 ***Construction :***

Let $u_0 \in V$ and define the sequence $(u_n) \subset V$ as follows :

(a) The outer iteration defines the sequence

$$u_{n+1} = u_n + \tau_n p_n \quad (n \in \mathbb{N}) \qquad (4.2.1)$$

where $p_n \in V$ is the numerical solution of :

$$\langle F'(u_n)p_n, v\rangle_{H_0^2} = -\langle F(u_n) - b, v\rangle_{H_0^2} \quad (v \in V) \qquad (4.2.2)$$

Further, $\delta_n > 0$ is constant satisfying $0 < \delta_n \le \delta_0 < 1$

$$\tau_n = \min\left\{1, \frac{(1-\delta_n)}{(1+\delta_n)} \ \frac{\mu_1}{L\|p_n\|_{H_0^2}}\right\} \in (0,1] \qquad (4.2.3)$$

(b) To determine p_n in (4.2.2) , the inner iteration defines a sequence

$$\left(p_n^{(k)}\right) \subset V \qquad (k \in \mathbb{N})$$

using a preconditioned conjugate gradient method .Here we have :

$$\mu_1|\varepsilon|_F^2 \le \left\langle \frac{\partial f}{\partial \theta}(x, D^2u_n(x))\varepsilon, \varepsilon \right\rangle \le \mu_2|\varepsilon|_F^2 \qquad \forall x \in \Omega, \varepsilon \in \mathbb{R}^N$$

Let $B{:}\,V \to \mathrm{V} \qquad \langle Bh, v\rangle_{H_0^2} = \int_\Omega D^2h \,.\, D^2v \qquad (h, v \in V)$

Then we consider the preconditioned form of (4.2.2) :

$$B^{-1}F'(u_n)p_n = -B^{-1}(F(u_n) - b)$$

Finally, $p_n := p_n^{(k_n)} \in V$ for which

$$\left\|F'(u_n)p_n^{(k_n)} + (F(u_n) - b)\ \right\|_{B_n^{-1}} \le \rho_n\|F(u_n) - b\|_{B^{-1}}$$

with $\rho_n = ({}^{\mu_1}/_{\mu_2})^{1/2}\delta_n$ and $\delta_n > 0$

4.3 Theorem :

Let Assumptions (4.1) be satisfied. Then construction yields the following convergence results :

(1) The outer iteration (u_n) satisfies

$$\|u_n - u^*\|_{H_0^2} \le \mu_1^{-1}\|F(u_n) - b\|_{H_0^2} \to 0 \text{ monotonically}$$

with speed depending on the sequence (δ_n) up to locally quadratic order. Namely, if $\delta_n \equiv \delta_0 < 1$, then the convergence is linear.

Further, if $\delta_n \le const. \|F(u_n) - b\|_{H_0^2}^{\gamma}$

with some constant $0 < \gamma \le 1$, then the convergence is locally of order $1 + \gamma : \|F(u_{n+1}) - b\|_{H_0^2} \le c_1\|F(u_n) - b\|_{H_0^2}^{\gamma+1} \qquad (n \ge n_0)$

yielding also the convergence estimate of weak order $1 + \gamma$:

$$\|F(u_n) - b\|_{H_0^2} \le d_1 q^{(1+\gamma)^n} \qquad (n \in \mathbb{N})$$

with suitable constants $0 < q < 1, d_1 > 0$

(2) there holds

$$\text{cond}(B^{-1}F'(u_n)) \leq \frac{\lambda_2}{\lambda_1}$$

And, accordingly, the inner iteration satisfies

$$\left\| F'(u_n)p_n^{(k_n)} + (F(u_n) - b) \right\|_{B^{-1}} \leq \left(\frac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\sqrt{\lambda_2} + \sqrt{\lambda_1}} \right)^k \|F(u_n) - b\|_{B^{-1}}$$

Therefore, the number of inner iterations for the nth outer step is at most

$k_n \in \mathbb{N}$ $$\left(\frac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\sqrt{\lambda_2} + \sqrt{\lambda_1}} \right)^{k_n} \leq \rho_n$$

with $\rho_n = \left({}^{\mu_1}/_{\mu_2}\right)^{1/2} \delta_n$

Let $\overline{g}$ be real function that is C^2 in the variable r , and there exists $\lambda_1, \lambda_2, \lambda > 0$ such that

$$0 < \lambda_1 \leq \overline{g}(r) \leq \lambda_2$$
$$0 < \lambda_1 \leq (\overline{g}(r^2)r)' \leq \lambda_2$$
$$\left| \frac{\partial^2}{\partial r^2} (\overline{g}(r^2)r \right| \leq \lambda \qquad (r \geq 0)$$

If these conditions hold for " $\overline{g}$ ", then the conditions of Theorem (4.3) are satisfied for the plate problem

$$\begin{cases} div^2(\overline{g}\big(E(D^2u)\big)\widetilde{D}^2u) = \alpha(x) \\ u_{|\partial\Omega} = \frac{\partial u}{\partial v}|_{\partial\Omega} = 0 \end{cases}$$

Hence the inner-outer method works for our model problem.

References :

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